教室ガール
2
茉崎ミユキ

MI.. MIRUKA-SAN...
STAY LIKE THIS... DON'T MOVE.

# 数学ガール

原作・結城浩

2

茅崎ミユキ

6th Period: The Small World of Our Making

Mathematical Girls
Godel's Incompleteness
Theorems

《ゲーデルの不完全性定理》

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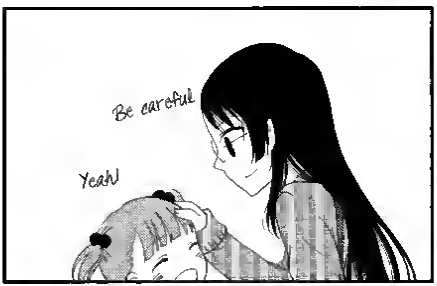
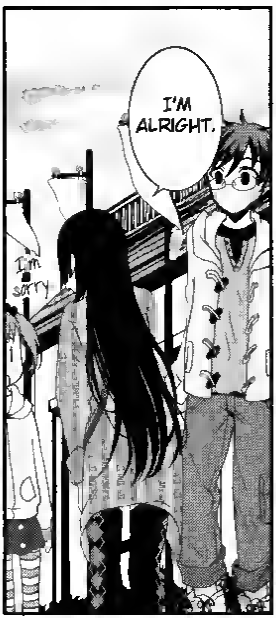
GOOD MORNING, MIRUKA-SAN!
First time seeing you in casual clothes ♡
GOOD MORNING.
WAH, MIRUKA-SAN IS CUTE!
RIGHT, ONII-CHAN?
HUH?
Waa
EH...
WELL,

Kyah!
BAM
fall
ARE YOU ALRIGHT, MIRUKA-SAN?


Be careful
Yeah!


I'M ALRIGHT.
I'm sorry

CAFE
THERE'S A MAP FOR THIS PLACE IN THE LEAFLET!

Kyaa
Kyaa
NOW THEN. I, TETRA, SHALL MAKE A SUGGESTION.
LET'S START WITH THIS 233-SECOND DREADFUL EXPERIENCE!
W-WE'RE GOING TO RIDE THIS TO START?
!?
VICTORY GOES TO THE SWIFTEST.
Clench

MI-MIRUKA-SAN...
HASN'T TETRA BEEN PROACTIVE LATELY...
THIS IS SUSPICIOUS ...
THAT'S
THAT'S NOT FAIR, ONII-CHAN!
IF IT'S TETRA, IT'S NOT YURI.

ONII-CHAN, YURI LIKES THIS ONE!
COFFEE CUPS?
Spin!
Spin!
Whirl
Whirl
Whoosh
Whoosh
!
I...I'M NOT REALLY GOOD WITH THINGS THAT ROTATE ..
I get sick..
K
ya
ha
ha
ha
ha
ha
K
ya
ha
ha
ha
ha
ha

Yuri
Aww
I got sick.
Wobble
Hey! Are you okay.
Akaha Onnichan. Yuri is dizzy
IF IT'S YURI, IT'S NOT TETRA.

This really looks like a Klein bottle...
NOT YET, LET'S ADJUST THE CURVATURE IN THIS AREA A BIT MORE...
IS IT OKAY LIKE THIS...
KIDS AREA
They're making something I don't understand...
HEY! THAT'S NOT FAIR, YURI-CHAN ME TOO...
YURI WILL GO FOR A WHILE AND BUY SOMETHING BECAUSE SHE IS THIRSTY...

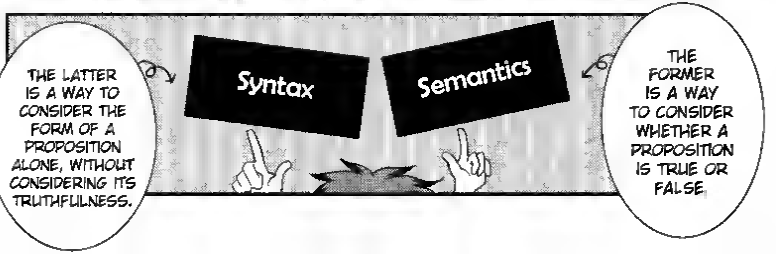
The Miruka-san who unravels complicated mathematics, and the Miruka-san who is eating soft ice-cream in front of me right now, are actually both the same Miruka-san.

...I'M NEITHER CUTE LIKE TETRA, NOR GENUINE LIKE YURI.
ONII-CHAN, NO FAIR!!
MUNCH MUNCH MUNCH
AH—!
THAT'S...

WHAT ICE CREAM DID YOU SENPAI GET?
It's already empty.
ONII-CHAN, YOU TWO HAVE ALREADY BEEN PLAYING WITH BLOCKS FOR NEARLY AN HOUR. YURI'S HAD ENOUGH.
One bite...
TIRAMISU, I THINK... PROBABLY...
?
PROBABLY?

HMM, IN A SENSE, THIS AMUSEMENT PARK IS SIMILAR TO A FORMAL SYSTEM
AS EXPECTED, CHOCOLATE FOR MIRUKA-SAN.

YES, WE TALKED THE OTHER DAY ABOUT BEING ABLE TO MATHEMATICISE MATHEMATICS IF WE USE SETS AND LOGIC...
Well
THAT'S RIGHT... THERE ARE TWO WAYS TO STUDY LOGIC, THE SEMANTIC WAY AND THE SYNTACTIC WAY.
YOU... DO YOU KNOW?


Syntax
Semantics
THE LATTER IS A WAY TO CONSIDER THE FORM OF A PROPOSITION ALONE, WITHOUT CONSIDERING ITS TRUTHFULNESS.
THE FORMER IS A WAY TO CONSIDER WHETHER A PROPOSITION IS TRUE OR FALSE.

YOU MIGHT SAY, THIS IS LIKE MAKING A WORLD OF YOUR OWN.
WITHOUT CONSIDERING THE TRUTHFULNESS OF PROPOSITIONS, YOU SAY . IS THERE SOMETHING LIKE THAT?
Mmm~
OF COURSE.
IN THE SYNTACTIC APPROACH, WE MAKE OURSELVES A FORMAL SYSTEM TO STUDY.
WE MAKE A FORMAL SYSTEM ON OUR OWN...?
BY SPECIFYING RULES THAT DEAL WITH FORM ONLY,
WITHOUT TAKING INTO CONSIDERATION ANY MEANINGS OR TRUTHFULNESS INVOLVING WORDS OTHER THAN THOSE IN THIS WORLD,
IT IS A WORLD THAT EXPANDS ONLY BY STRICT APPLICATION OF THOSE RULES.

IT RESEMBLES A THEME PARK.
IT'S THIS KIND OF A WORLD.
A WORLD THAT EXPANDS BY STRICT APPLICATION OF PRECISELY DEFINED RULES...
ISN'T THIS WHAT MATHS IS!?
EXACTLY, IN A SENSE; DEALING WITH MATHEMATICS AS A FORMAL SYSTEM IS AN OBJECTIVE.
IN THAT CASE, ANALYSING MATHEMATICS ITSELF INSIDE THE FORMAL SYSTEM IS POSSIBLE.

Formation rule F1
If x is a variable, x is a proposition
Formation rule F2
If x is a logical equation, ¬(x) is also a proposition
Formation rule F3:
If x and y are logical equations, (x)∨(y) is also a proposition
Formation rule F4
Propositions are exactly the strings defined by rules F1–F3.

Symbol IMPLIES: (x)→(y) is defined to be (¬(x))∨(y)

Axiom P1 ((x)∨(x))→(x)
Axiom P2 (x)→((x)∨(y))
Axiom P3 ((x)∨(y))→((y)∨(x))
Axiom P4 ((x)→(y))→(((z)∨(x))→((z)∨(y)))

Provided that x, y, z represent arbitrary propositions
The axioms are always valid

Rule of inference Modus Ponens: If x and (x)→(y) have been written earlier, it is valid to write y; here x and y represent arbitrary propositions

SWISH

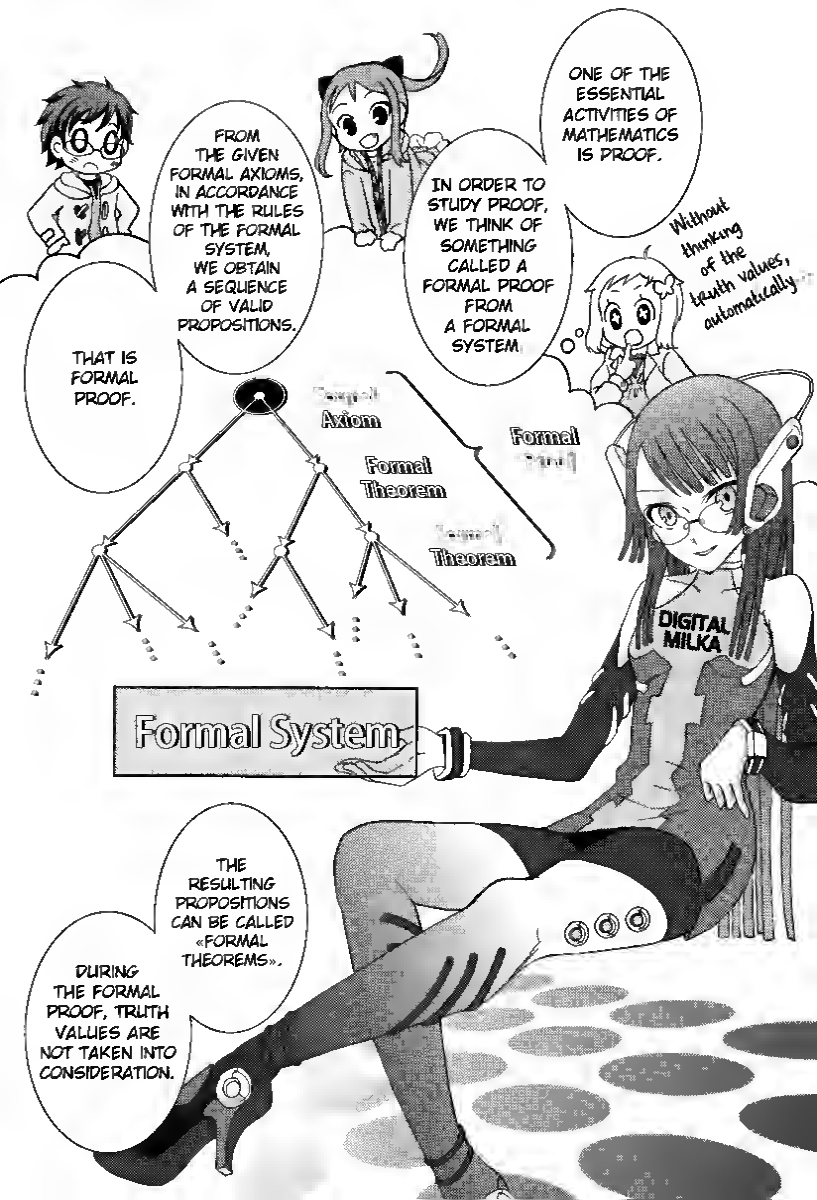
This is an example of a formal system

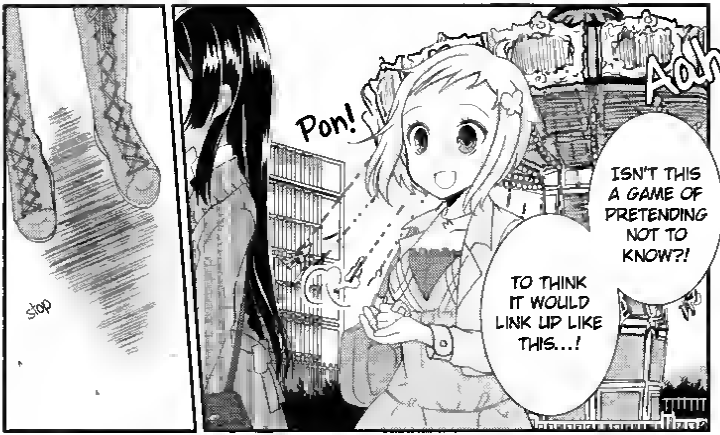
BUT STILL, WHY BOTHER TO CREATE A FORMAL SYSTEM AND THINK INSIDE IT?

THE BASIS OF A LOGICAL ARGUMENT INDEPENDENT OF THE MEANINGS OF THE PROPOSITIONS BECOMES CLEAR. THUS IT IS POSSIBLE TO AUTOMATICALLY ADVANCE THE DISCUSSION

Thank you for the food

I'm really full


ONE OF THE ESSENTIAL ACTIVITIES OF MATHEMATICS IS PROOF.
IN ORDER TO STUDY PROOF, WE THINK OF SOMETHING CALLED A FORMAL PROOF FROM A FORMAL SYSTEM.
Without thinking of the truth values, automatically
FROM THE GIVEN FORMAL AXIOMS, IN ACCORDANCE WITH THE RULES OF THE FORMAL SYSTEM, WE OBTAIN A SEQUENCE OF VALID PROPOSITIONS.
THAT IS FORMAL PROOF.
Axiom
Formal Theorem
Formal Theorem
Formal System
DIGITAL MILKA
THE RESULTING PROPOSITIONS CAN BE CALLED «FORMAL THEOREMS».
DURING THE FORMAL PROOF, TRUTH VALUES ARE NOT TAKEN INTO CONSIDERATION.


Stop
Pon!
Aah
ISN'T THIS A GAME OF PRETENDING NOT TO KNOW?!
TO THINK IT WOULD LINK UP LIKE THIS...!

WHY DON'T WE ALL RIDE THIS LAST?

FIRST IS ROCK-!
THAT'S EXACTLY WHAT I WANT.
TETRA-SAN. THE SEATING ORDER SHALL BE DETERMINED FAIR AND SQUARE BY ROCK PAPER SCISSORS
IT'S A TIE-
NEXT CUSTOMER-
16

whoosh
NUMBER 17- A BEAUTIFUL PRIME NUMBER
IT'S A TIE-
17
NOW, WHAT WILL YOU DO?

Guon Guon
I WON!!!
I LOST~
HUH. UWAAAAA-!! THEY'RE ALREADY GONE~

WAH... PEOPLE ARE GETTING GRADUALLY SMALLER...
AS IF THEY WERE MINIATURES.

exhausted
I WONDER, AREN'T WE GAZING AT A FORMAL WORLD RIGHT NOW?
WHAT'S WRONG? ARE YOU FEELING SICK?
IT'S NOTHING, JUST DON'T MOVE.
IDIOT! DON'T STAND!
ARE YOU ALRIGHT?

ONII-CHAN STOOD UP?!
WH...WHAT'S GOING ON?
TH...THEY GOT OUT OF SIGHT~
clunk clunk
Onii-chan~!! Senpai~!!

MIRUKA-SAN, ARE YOU ALRIGHT?!

...WHAT IF IT FALLS...
DON'T SHAKE...
SORRY!
release
Could it be that Miruka-san is no good with high places?
SORRY, I'LL RETURN TO MY SEAT NOW.
IF YOU RETURN, IT WILL PROBABLY SHAKE AGAIN!
MI... MIRUKA-SA...
DON'T... MOVE!
DON'T MOVE! DON'T MOVE!

STAY LIKE THIS... DON'T MOVE.
clutch
I, TOO, HAVE THINGS THAT I FIND SCARY...
IF YOU'RE SO BAD WITH THIS, DON'T PUSH YOURSE...
IT'S NOT ABOUT THE FERRIS WHEEL.

Even though I don't understand, Miruka-san somehow feels a little bit closer than usual...

touch
WELL, MIRUKA-SAN.
IT'S REALLY ALRIGHT.
If it's Miruka-san, it's Miruka-san.
sigh...

17

Ouu
LET'S TALK ABOUT MATHS AGAIN.
MORE AND MORE CAREFULLY
YES.

# 6th Period

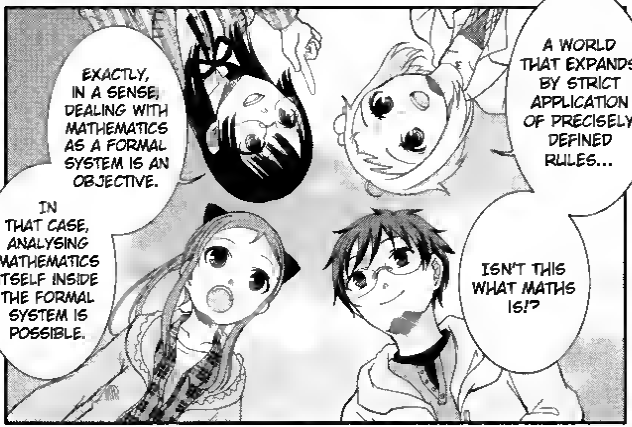
# Formal System

Mathematics makes various things into subjects of study.
Surprisingly, even proof, which takes place in mathematics, can be made into such a subject. The formal system which was introduced in this 6th period is one of the tools that can be utilized in order to study proof.

Although formal logic is a topic studied in mathemabcal logic at universities, it is a very modern theme of mathematics. That's because what can be achieved by dealing formally with strings of characters and symbols without considering their meaning holds great significance for computers.

It's not correct to restrict computers to "machines that perform mathematical calculations". Modern computers provide us with help by manipulating symbols and unraveling complex logic.

As it appears in this issue, formal systems are not studied in middle or high school. But, if you go to libraries or book stores, there will be a lot of maths books on these subjects. Those who have an interest or are ambitious, how about standing on your toes, stretching yourselves and trying to challenge those books?


A WORLD THAT EXPANDS BY STRICT APPLICATION OF PRECISELY DEFINED RULES...
ISN'T THIS WHAT MATHS IS!?
EXACTLY, IN A SENSE, DEALING WITH MATHEMATICS AS A FORMAL SYSTEM IS AN OBJECTIVE.
IN THAT CASE, ANALYSING MATHEMATICS ITSELF INSIDE THE FORMAL SYSTEM IS POSSIBLE.

Mathematical Girls

Gödel's Incompleteness Theorems

# Mathematical Girls

C06

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## 7th Period: Chocolate Verifier

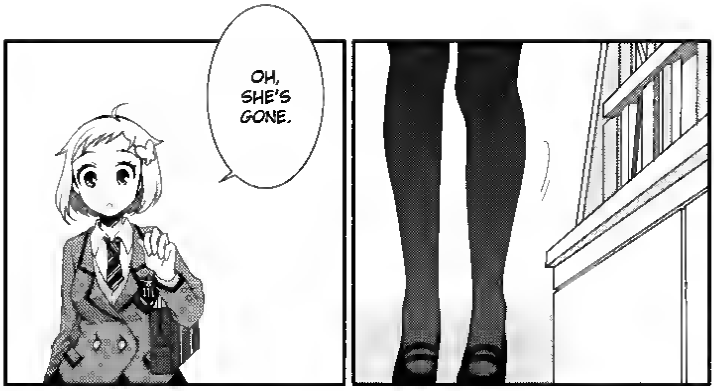
February 13

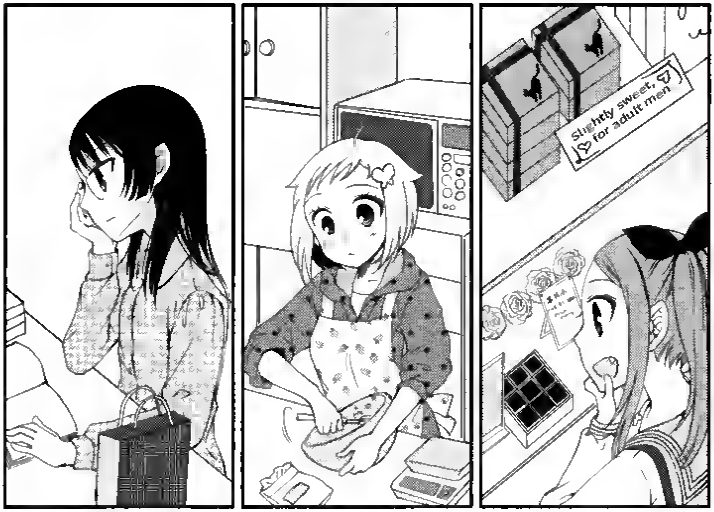
twitch
WELL, BEING A POPULAR LADY HAS ITS OWN EXPENSES.
HEY YOU, BUYING LOTS OF FINE CHOCOLATES AGAIN, HUH!
St. Valentines Day
I WONDER IF THERE'S COFFEE FLAVOUR OR ANYTHING LIKE THAT...
ONII-CHAN HAS BEEN CALLING COCOA CHILDISH...
YURI... IS NOT A CHILD ANYMORE!! ONE PIECE SHOULD BE AFFORDABLE!
Peek
Click!

VANISH
(Reflexively)
MI-MIRUKA-SAMA IS HERE TO BUY VALENTINE'S CHOCOLATES AS WELL?!
WHO MIGHT BE THE ONE SHE'LL GIVE THEM TO...?!

B
B-01
B-02
WAH! I LOST SIGHT OF HER.
Hmm
A
TOEIC Test

H..HM
?
Mi
stare...
B-01
English
MIRUKA-SAMA IS LOOKING AT BOOKSHELVES NOT RELATED TO MATHS...




February

# 14

tra-
clump
tra-
clump
tra-tra-
tra-tra-
GOOD MORNING!
MORNING, TETRA-CHAN. AREN'T YOU EARLY TODAY.
OH, UH, YES!

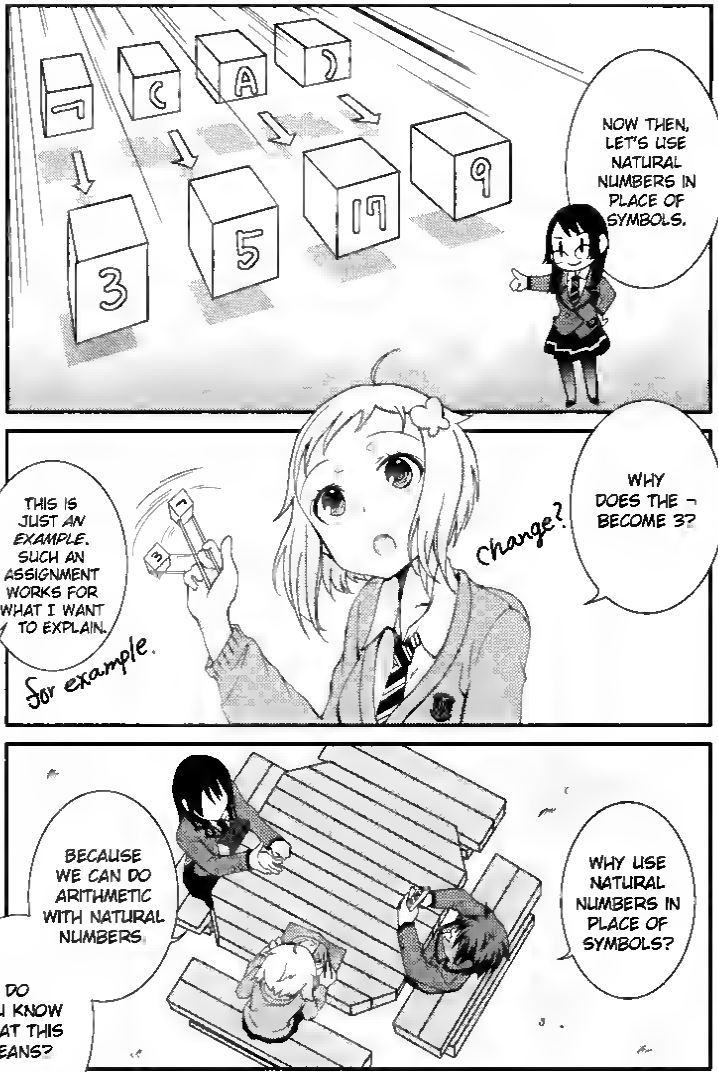
SE-SENPAI...!
WHAT?
TODAY IS A REALLY AUSPICIOUS DAY~
Unknown
rustle rustle
YES, WHAT NICE WEATHER
Squeeze
NO, UM...
ding dong
WELL, UM...
EXTEND
EH? THIS IS FOR ME?
BLUSH
falls
H-HERE!

THANK YOU, YURI WILL BE HAPPY.
turns
Y-YES!
PLEASE EAT IT TOGETHER WITH YURI-CHAN.
I auto-matically
Well then see you again at lunch!
Yes.
We have to hurry
The bell's rung.

TETRA, WHAT WAS A FORMAL SYSTEM?
U-UMM.
Ha !!
IT'S A SYSTEM WE CREATE IN ORDER TO THINK USING THE FORM OF PROPOSITIONS ALONE WITHOUT CONSIDERING WHETHER THEY ARE TRUE.
THAT'S RIGHT, AND SYMBOLS ARE THE ROOT OF THIS.

IS THIS ABOUT THE VARIOUS SYMBOLS THAT PROPOSITIONS ARE MADE UP OF?
Like these

RIGHT, BUT THOSE SYMBOLS DON'T HAVE TO LOOK LIKE THAT.
I SEE. IT DOESN'T MATTER IF WE ALTER THEM IN ANY WAY CONVENIENT FOR US!
SINCE WE ARE THINKING BY FORM ONLY, IT ENDS UP LIKE THIS.


NOW THEN, LET'S USE NATURAL NUMBERS IN PLACE OF SYMBOLS.
¬
(
A
)
3
5
17
9
WHY DOES THE ¬ BECOME 3?
change?
THIS IS JUST AN EXAMPLE. SUCH AN ASSIGNMENT WORKS FOR WHAT I WANT TO EXPLAIN.
for example.
WHY USE NATURAL NUMBERS IN PLACE OF SYMBOLS?
BECAUSE WE CAN DO ARITHMETIC WITH NATURAL NUMBERS.
DO YOU KNOW WHAT THIS MEANS?

!
nod
Hmm
IN OTHER WORDS?

THE FORMAL SYSTEM CAN BE CONSIDERED ARITHMETICALLY ...RIGHT?

Arithmetic

Natural Numbers

Symbols

Formal System

We assign appropriate natural numbers for the various symbols

For example

| ¬ | ( | A | ) | ⇒ | 3 | 5 | 17 | 9 |
|---|---|---|---|---|---|---|---|---|

By placing them as exponents for the sequence of prime numbers

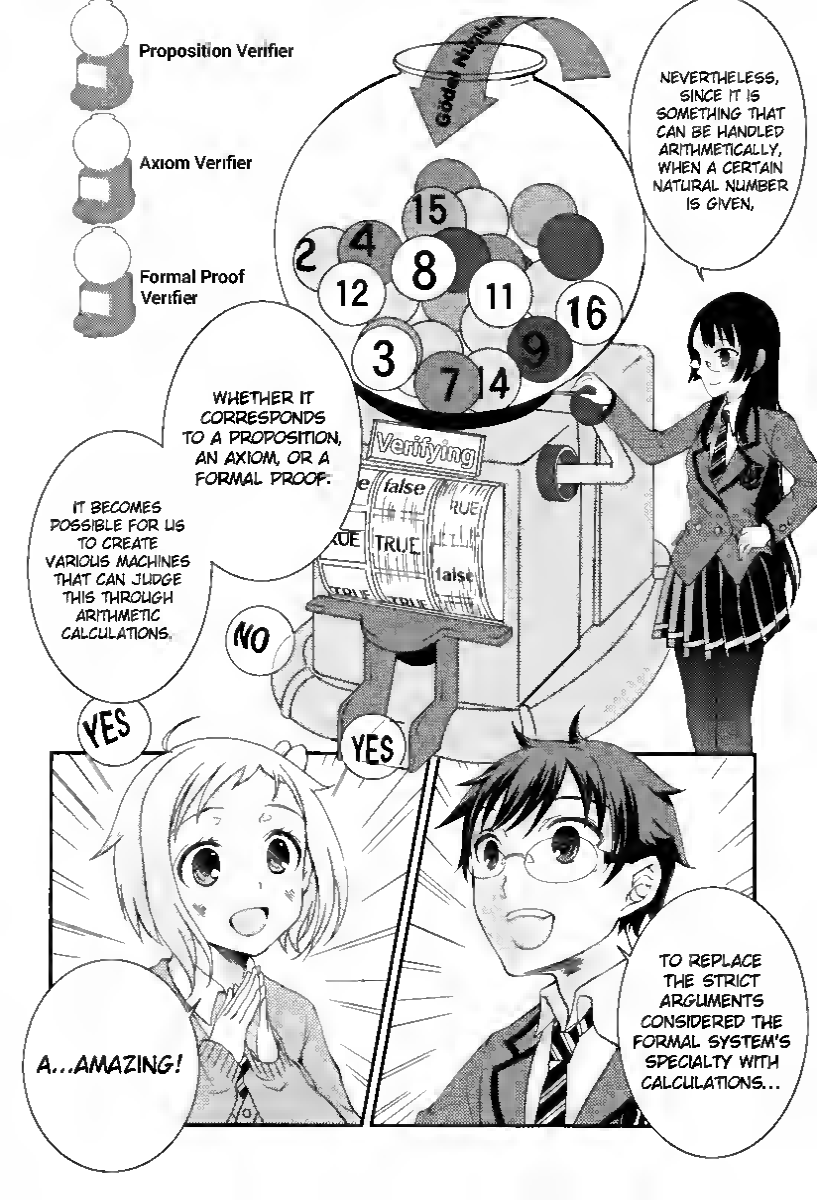
$\langle 2, 3, 5, 7 \cdots \rangle$

$$2^{\neg} \times 3^{(} \times 5^{A} \times 7^{)}$$
$$= 2^{3} \times 3^{5} \times 5^{17} \times 7^{9}$$
$$= 8 \times 243 \times 762939453125 \times 40353607$$
$$= 598506256164550\,78125000$$

It is possible to create a «unique sequence of digits» from any proposition

We call this a Gödel number.

I SEE, SO BY USING THE QUALITIES OF PRIME NUMBERS, GÖDEL NUMBERS AND PROPOSITIONS CAN BE INTER-CONVERTED FREELY, HUH


Proposition Verifier
Axiom Verifier
Formal Proof Verifier
Gödel Numbering
15
2
4
8
12
11
16
3
9
7
14
Verifying
false
TRUE
TRUE
false
NEVERTHELESS, SINCE IT IS SOMETHING THAT CAN BE HANDLED ARITHMETICALLY, WHEN A CERTAIN NATURAL NUMBER IS GIVEN,
WHETHER IT CORRESPONDS TO A PROPOSITION, AN AXIOM, OR A FORMAL PROOF.
IT BECOMES POSSIBLE FOR US TO CREATE VARIOUS MACHINES THAT CAN JUDGE THIS THROUGH ARITHMETIC CALCULATIONS.
NO
YES
YES
A...AMAZING!
TO REPLACE THE STRICT ARGUMENTS CONSIDERED THE FORMAL SYSTEM'S SPECIALTY WITH CALCULATIONS...

DOING MATHS WITH EVERYONE REALLY IS FUN.
BREEZE
LET'S DO MATHS AGAIN TOMORROW!!
Senpai! Miruka-san!!
A...AH.
MIRUKA-SAN?

Rustle
HEY YOU, I'M GIVING YOU THIS.
Th-that's...
TH... THANK YOU.
I wonder what it is.

AND THAT WAS WHAT WE WERE TALKING ABOUT AT LUNCH.
VERIFICATION, HUH- THAT'S COOL!
DO YOU REMEMBER, TALKING ABOUT DOING MATHS WITH A FORMAL SYSTEM AT THE AMUSEMENT PARK A WHILE AGO?
AND YET THE STORY DOESN'T END THERE.
EH...?
Y-YES.

EXACTLY.
EXPRESS THE FORMAL SYSTEM WITH ARITHMETIC, AND EXPRESS THAT ARITHMETIC USING A FORMAL SYSTEM
Could it be.
YOU DON'T SAY...THAT ARITHMETIC IS...?
WHICH MEANS?
A FORMAL SYSTEM OF A FORMAL SYSTEM

FU-
FU
AA
AA
AA!
BY CONCRETELY ESTABLISHING THIS, THE FIRST INCOMPLETENESS THEOREM WAS PROVED BY GODEL.
BUT YOU KNOW.. DISCUSSING MATHS IS AS AMAZING AS ALWAYS..
RECENTLY, MIRUKA-SAN HAS BEEN ACTING KIND OF STRANGE..
LIKE GETTING ABSENT-MINDED IN THE MIDDLE OF DISCUSSIONS AND READING DIFFERENT BOOKS FROM USUAL.
AND... GIVING ME THAT MYSTERIOUS PRESENT.

EEEH?
!
Pi-Pin!
Now that you mention it yesterday...

ISN'T THAT... LOVE?

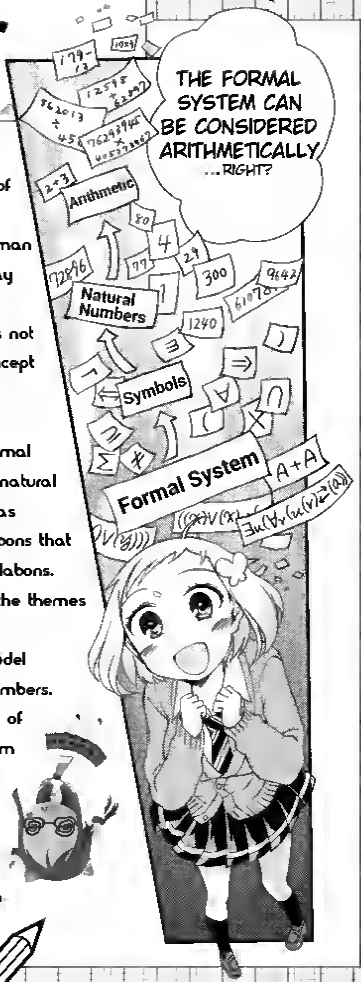
Even though I don't know who Miruka-sama gave hers to...
WELL...
rustle rustle
SINCE YURI IS AN ADULT ALREADY, FOR HER UNPOPULAR ONII-CHAN'S SAKE..
EH
bar dump
ONII-CHAN IS REALLY DENSE

At least I prepared chocolate
turns
ONII-CHAN
AH, COME TO THINK OF IT,
I WAS ENTRUSTED WITH A PRESENT FOR YURI FROM TETRA-CHAN
EH?! FROM TETRA-CHAN?!
Excited
Hard stare
FWOOSH
is there anything else...
Yu-Yuri!
THANK YOU, ONII-CHAN
HEY... ISN'T THIS CHOCOLATE !?
Handmade!?

YURI WON'T GIVE HERS TO ONII-CHAN!
rip
rip
Eh?!
Th-That ! piece fi-five hund...
munch
SNIFF
It was meant for onii-chan.
TH-THIS IS BITTER ~~~!
CACAO 99%
CACAO 99%
CACAO 99%
Extremely Bitter Chocolate
We recommend eating this by itself or drinking something sweet to accompany it.

# 7th period

# Verifier




When "machines" are mentioned, what sort of image comes to mind? Automatic, without a heart, bloodless and uncomprehending of human feeling. But they work for us in the same way every time. Probably this sort of image, right?

The "verifier" that appears in this 7th period is not an actual machine, it is a mathematical concept.

Since a formal system's propositions can be represented by symbols, if symbols can be represented by natural numbers, then the formal system's propositions can be represented by natural numbers. If we deal with these propositions as natural numbers, we can "verify" the propositions that those natural numbers express through calculations.

What can and can't be computed is one of the themes studied in the science of logic.

In 7th period, we learned how to make a Gödel number by expressing symbols as natural numbers. Nowadays it may not be expressed in terms of prime numbers; it may be expressed in a form that's easy for computers to deal with, right?

In our modern life, computers are essential.

It might be that precisely because machines work mechanically, they are able to help humans with the work they're not good with.

Mathematical Girls

Godel s Incompleteness Theorems

# Mathematical Girls

## c07

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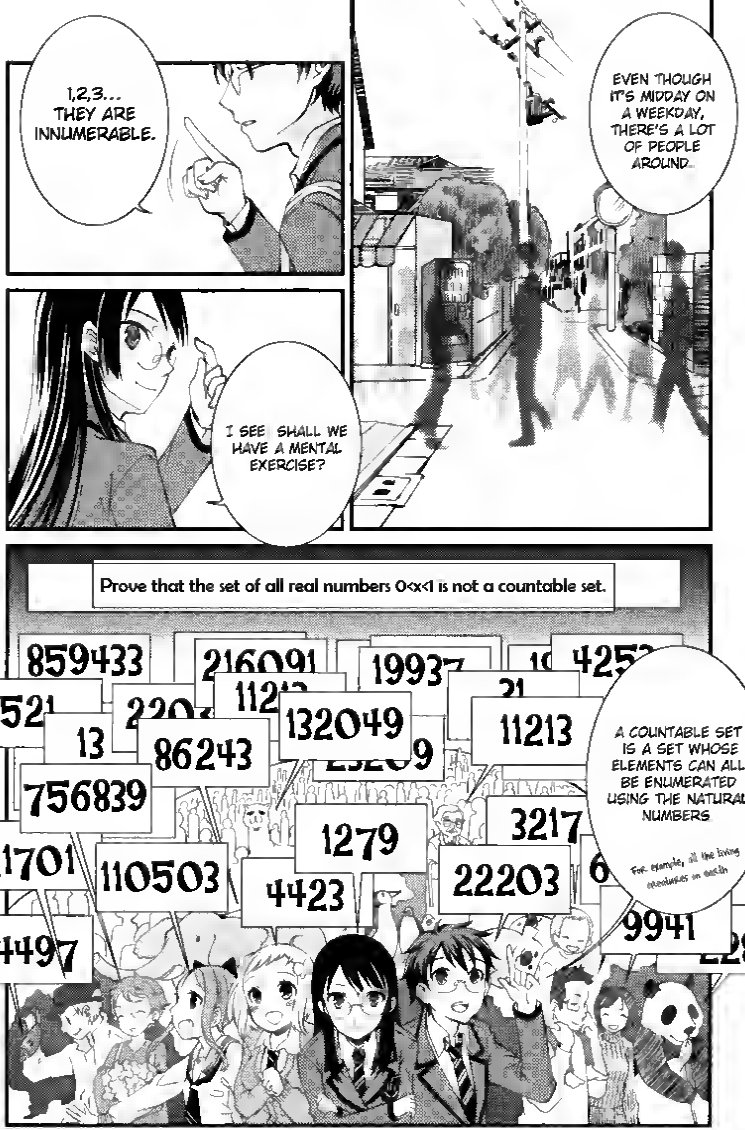
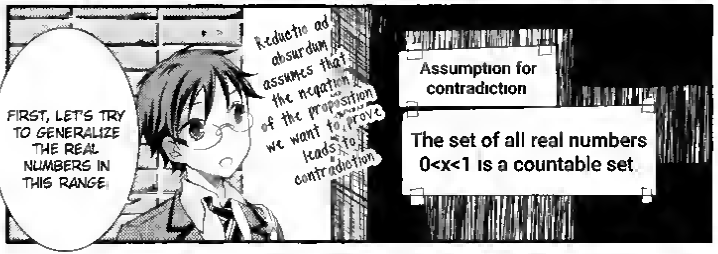
Even though it's actually time for class...
MIRUKA-SAN, WAIT!
WHERE ARE YOU GOING?
OURS WAS CANCELLED.
But—

THE WEATHER IS REALLY GOOD.
DOING SOMETHING DIFFERENT WITH EVERYONE ONCE IN A WHILE IS NICE
LET'S GO.

## 8th period: Continuing Up the Spiral

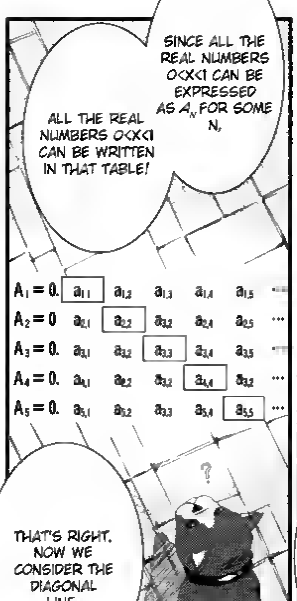

1,2,3... THEY ARE INNUMERABLE.
EVEN THOUGH IT'S MIDDAY ON A WEEKDAY, THERE'S A LOT OF PEOPLE AROUND...
I SEE SHALL WE HAVE A MENTAL EXERCISE?
Prove that the set of all real numbers 0<x<1 is not a countable set.
859433
521
13
86243
756839
21701
110503
44497
132049
11213
1279
4423
3217
22203
9941
A COUNTABLE SET IS A SET WHOSE ELEMENTS CAN ALL BE ENUMERATED USING THE NATURAL NUMBERS
For example, all the living creatures on earth

EXACTLY NOW THEN, HOW WILL YOU PROVE IT?
IN THIS KIND OF CASE, USING REDUCTIO AD ABSURDUM SOUNDS GOOD. LET ME SEE..


FIRST, LET'S TRY TO GENERALIZE THE REAL NUMBERS IN THIS RANGE.
Reductio ad absurdum assumes that the negation of the proposition we want to prove leads to a contradiction.
Assumption for contradiction
The set of all real numbers 0<x<1 is a countable set

In the $n^{th}$ real number $A_n$, let us call the digit at the $m^{th}$ decimal place $a_{n,m}$
$A_n = 0.\ a_{n,1}\ a_{n,2}\ a_{n,3}\ a_{n,4}\ a_{n,5}\cdots$
$A_5 = 0.\ 3\ 1\ 4\ 1\ 5\cdots$
$a_{5,1}\ a_{5,2}\ a_{5,3}\ a_{5,4}\ a_{5,5}\cdots$
FOR EXAMPLE, PERHAPS $A_5 = 0.31415..$
$a_{5,1} = 3$
$a_{5,2} = 1$
$a_{5,3} = 4$
$a_{5,4} = 1$
$a_{5,5} = 5\cdots$
LOOKING AT IT THIS WAY IS EASIER.

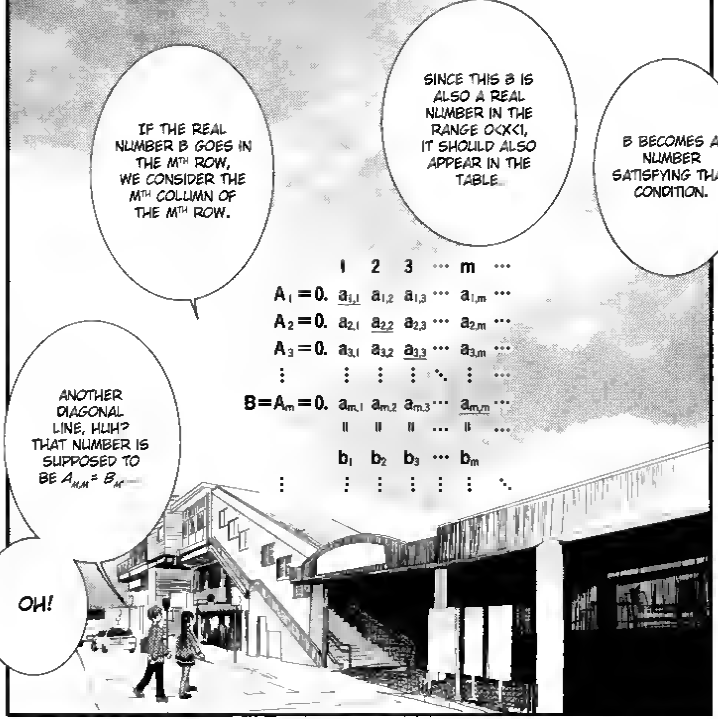
HAVE YOU ALREADY USED THE CONDITION THAT «IT IS A COUNTABLE SET»?
58830
58831
12801
58832
HMM, HOW DO WE DO IT FROM HERE...
18695
54637
702
AND THEN?
279
WELL, LET'S TRY ARRANGING THEM
SINCE TO SPEAK OF «THE N^TH REAL NUMBER A_x» FOR N = 1, 2, 3, ... THEY WERE NUMBERED USING THE NATURAL NUMBERS.
EH? ...YEAH, I USED IT.
DOESN'T THIS LOOK A BIT LIKE A TABLE?
$A_1 = 0.a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{1,4}\ a_{1,5}\cdots$
$A_2 = 0.a_{2,1}\ a_{2,2}\ a_{2,3}\ a_{2,4}\ a_{2,5}\cdots$
$A_3 = 0.a_{3,1}\ a_{3,2}\ a_{3,3}\ a_{3,4}\ a_{3,5}\cdots$
$A_4 = 0.a_{4,1}\ a_{4,2}\ a_{4,3}\ a_{4,4}\ a_{4,5}\cdots$
$A_5 = 0.a_{5,1}\ a_{5,2}\ a_{5,3}\ a_{5,4}\ a_{5,5}\cdots$
I SEE. SO IF IT WERE A COUNTABLE SET IT SHOULD BE POSSIBLE TO PUT IT IN ORDER


ALL THE REAL NUMBERS 0<X<1 CAN BE WRITTEN IN THAT TABLE!
SINCE ALL THE REAL NUMBERS 0<X<1 CAN BE EXPRESSED AS $A_N$ FOR SOME N,
$A_1 = 0.\ a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{1,4}\ a_{1,5}\ \cdots$
$A_2 = 0\ a_{2,1}\ a_{2,2}\ a_{2,3}\ a_{2,4}\ a_{2,5}\ \cdots$
$A_3 = 0.\ a_{3,1}\ a_{3,2}\ a_{3,3}\ a_{3,4}\ a_{3,5}\ \cdots$
$A_4 = 0.\ a_{4,1}\ a_{4,2}\ a_{4,3}\ a_{4,4}\ a_{4,5}\ \cdots$
$A_5 = 0.\ a_{5,1}\ a_{5,2}\ a_{5,3}\ a_{5,4}\ a_{5,5}\ \cdots$
THAT'S RIGHT. NOW WE CONSIDER THE DIAGONAL LINE.

NOW THAT YOU SAY SO!

WELL THEN, LET'S ARRANGE ALL THE $A_N$ AND PUT THEM IN A TABLE.
WHAT KIND OF PROPERTIES DOES THAT TABLE HAVE?

When $a_{n,n}$ is even, $b_n = 1$
When $a_{n,n}$ is odd, $b_n = 2$
then:
$B = 0.b_1b_2b_3b_4b_5\cdots$
$a_{n,n} \neq b_n$
STARTING WITH THIS REAL NUMBER, LET'S CREATE ONE MORE REAL NUMBER WITH THE CONDITION THAT EACH DIGIT IS DIFFERENT.
$a_{1,1}\ a_{2,2}\ a_{3,3}\ a_{4,4}\ a_{5,5}\cdots$
THIS CAN BE THOUGHT OF AS A REAL NUMBER IN THE SAME WAY.
$0.a_{1,1}\ a_{2,2}\ a_{3,3}\ a_{4,4}\ a_{5,5}\cdots a_{k,k}\cdots$


IF THE REAL NUMBER B GOES IN THE Mᵀᴴ ROW, WE CONSIDER THE Mᵀᴴ COLUMN OF THE Mᵀᴴ ROW.
SINCE THIS B IS ALSO A REAL NUMBER IN THE RANGE 0<X<1, IT SHOULD ALSO APPEAR IN THE TABLE.
B BECOMES A NUMBER SATISFYING THAT CONDITION.
1 2 3 ⋯ m ⋯
A₁ = 0. a1,1 a1,2 a1,3 ⋯ a1,m ⋯
A₂ = 0. a2,1 a2,2 a2,3 ⋯ a2,m ⋯
A₃ = 0. a3,1 a3,2 a3,3 ⋯ a3,m ⋯
B=Am = 0. am,1 am,2 am,3 ⋯ am,m ⋯
b1 b2 b3 ⋯ bm
ANOTHER DIAGONAL LINE, HUH? THAT NUMBER IS SUPPOSED TO BE $A_{n,n} = B_n$.
OH!

smile
BY REDUCTIO AD ABSURDUM, THE SET OF ALL THE REAL NUMBERS X SUCH THAT 0<X<1 IS NOT A COUNTABLE SET.
THIS CONTRADICTS $A_{n,n} = B_n$, DOESN'T IT?
WELL DONE.
WHICH MEANS...

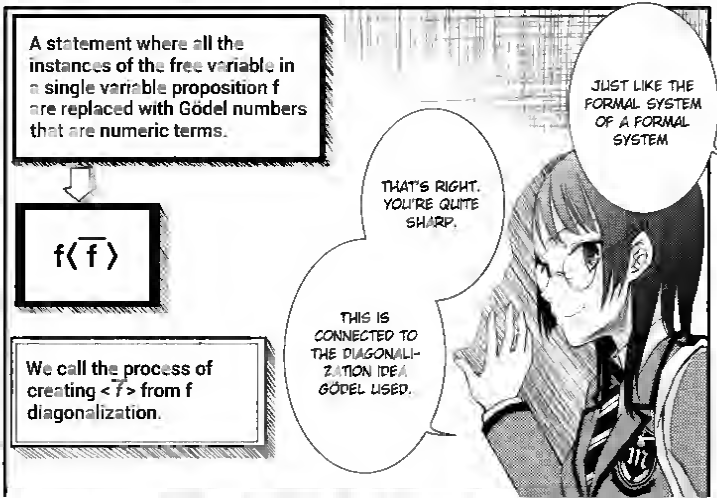
WAIT!
PRRRR
PLATFORM 3! THE TRAIN IS COMING!
PSSH
Next is ... station
RIP
THE TRAIN IS EMPTY

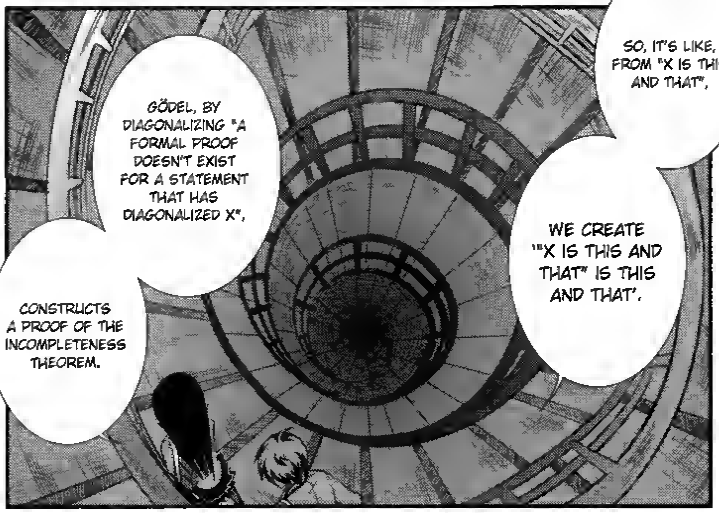
Chocl...
ISN'T THAT... LOVE?
YOU TOO, WANNA EAT SOME?
A- YEAH
THANKS.

Grab
JR
THIS WAY.

I WONDER, COULD THE GRAINS OF SAND ON A BEACH BE COUNTABLE?
FUFU, HOW ABOUT TRYING TO PROVE IT?
OVER THERE.

THE DISCUSSION WE HAD A WHILE AGO WAS AMAZING! RECURSIVELY APPLYING SOMETHING WE CREATED OF OUR OWN ACCORD TO OURSELVES...
THIS RECURSIVE SELF-REFERENCE IS VERY INTERESTING


A statement where all the instances of the free variable in a single variable proposition f are replaced with Gödel numbers that are numeric terms.
f⟨f̄⟩
We call the process of creating ⟨f̄⟩ from f diagonalization.
JUST LIKE THE FORMAL SYSTEM OF A FORMAL SYSTEM
THAT'S RIGHT. YOU'RE QUITE SHARP.
THIS IS CONNECTED TO THE DIAGONALIZATION IDEA GÖDEL USED.

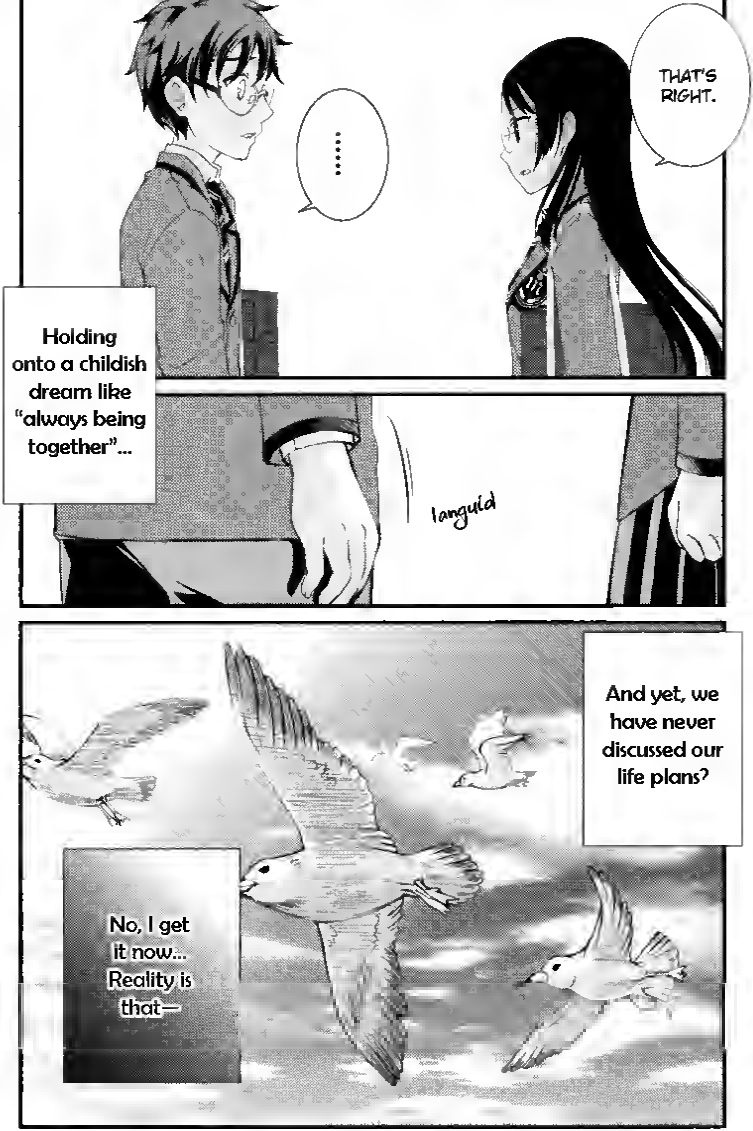

SO, IT'S LIKE, FROM "X IS THIS AND THAT",
WE CREATE '"X IS THIS AND THAT" IS THIS AND THAT'.
GÖDEL, BY DIAGONALIZING "A FORMAL PROOF DOESN'T EXIST FOR A STATEMENT THAT HAS DIAGONALIZED X",
CONSTRUCTS A PROOF OF THE INCOMPLETENESS THEOREM.

A formal proof doesn't exist for a statement that has diagonalized "a formal proof doesn't exist for a statement that has diagonalized X".

FWOOSH
HEY, YOU...
bardump bardump

I HAVE AN OFFER TO STUDY ABROAD AT AN AMERICAN UNIVERSITY.
...AN OFFER-
EH...

THAT'S- HAVE YOU ALREADY DECIDED?
sweat
Eh... so suddenly, even though I'm not mentally prepared,
I'VE DECIDED
WILL YOU BE GOING OVERSEAS?
The presentiment of going to the same university with Miruka-san once we graduate high school is...

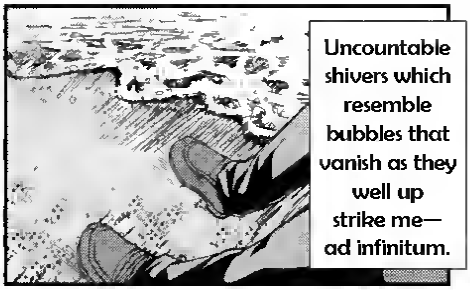

THAT'S RIGHT.
.....
Holding onto a childish dream like "always being together"...
languid
And yet, we have never discussed our life plans?
No, I get it now... Reality is that—

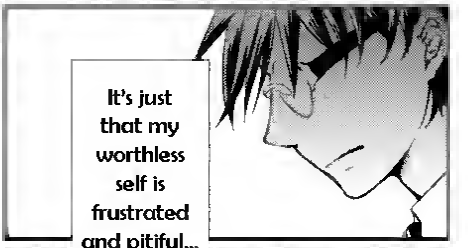
...I DIDN'T KNOW.
HUH?
"Soon we will— —be apart"
THIS MUST HAVE BEEN COMING ABOUT FOR A WHILE.

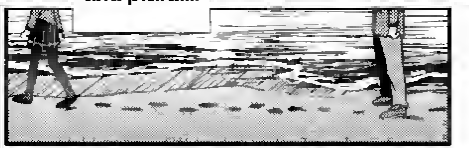
YET UP TILL NOW I HAVEN'T KNOWN ANYTHING ABOUT IT AT ALL.
NOTHING ABOUT THE COURSE OF MIRUKA-SAN'S LIFE.
.....
I— what the heck am I saying
DOES THIS MEAN THAT YOU DON'T TRUST ME?
THAT'S NOT TRUE, I...
Miruka-san's words are no longer entering my ears.

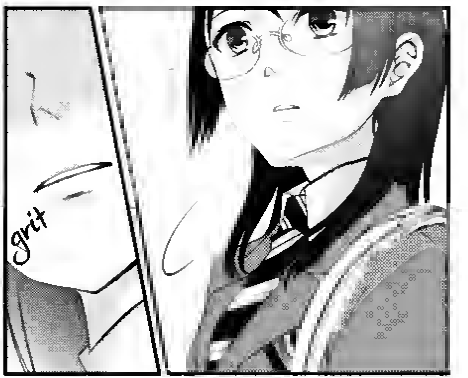
SHAA
I'm— just a kid.
Even though an important person to me is leaving, I haven't been able to say anything... other than snide comments.

step
step
step
step


Uncountable shivers which resemble bubbles that vanish as they well up strike me—ad infinitum.


It's just that my worthless self is frustrated and pitiful...

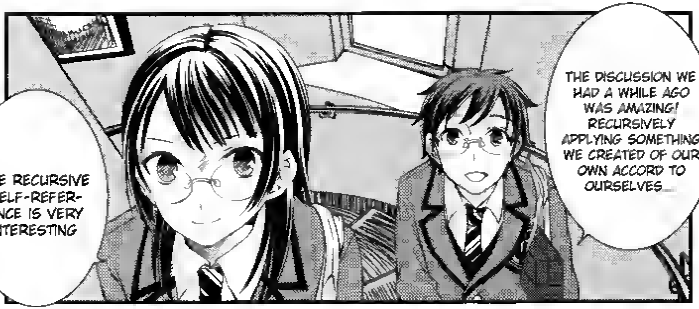



Grit

SL
AD
IDIOT!

# 8th period

# Diagonal Reasoning


THE RECURSIVE SELF-REFERENCE IS VERY INTERESTING
THE DISCUSSION WE HAD A WHILE AGO WAS AMAZING! RECURSIVELY APPLYING SOMETHING WE CREATED OF OUR OWN ACCORD TO OURSELVES...

In this 8th period we studied Cantor's diagonal reasoning. The proof that real numbers are uncountable via diagonal reasoning is very famous. But it's a story that makes one feel deceived. That's because dealing with what's called "infinity" is a very difficult thing. Usually we never think that infinity has types or the like. It is unthinkable that infinity is anything but infinity. However, using Cantor's diagonal argument, the difference between infinities that can be enumerated (countable sets), and infinities that can't be enumerated (uncountable sets) can be carved out. The natural numbers, integers, rational numbers, etc. are all countable sets. On the other hand, the real numbers or complex numbers are both uncountable sets. Dealing with the concept called infinity is a very important matter in the study of mathematics. Using the words we use every day, accurately discussing types of infinity is extremely difficult.

If we don't use numerical or logical formulas, the "words of mathematics", we can't deal with infinity precisely. Although there are many people who dislike numerical or logical formulas, please think of them as wonderful words that can deal with even that mysterious entity called infinity.

Mathematical Girls
Godel's Incompleteness
Theorems

# Mathematical Girls

C08

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Illustrator スらんと

David Hilbert—
A leading mathematician who played an active role in the 19th century and the beginning of the 20th century.
TO ESTABLISH A SOLID FOUNDATION FOR MATHEMATICS, HE PROPOSED A CERTAIN PROGRAM ITS NAME IS THE HILBERT PROGRAM
The three steps of the Hilbert program:
① Introducing a formal system
Expressing mathematics as a «formal system»
② Proof of consistency
Proving that the mathematics expressed as a formal system is «consistent»
③ Proof of completeness
Proving that the mathematics expressed as a formal system is «complete»
I UNDERSTAND WHAT'S MEANT BY EXPRESSING MATHEMATICS AS A «FORMAL SYSTEM». BUT
WHAT DO YOU MEAN BY «CONSISTENT» AND «COMPLETE»?
FIRST OF ALL, WHAT «CONSISTENT» MEANS IS

FOR ANY PROPOSITION A IN THE LANGUAGE OF THIS FORMAL SYSTEM,
«THERE CAN NEVER BE A FORMAL PROOF OF BOTH A AND ¬A», THAT'S WHAT IT MEANS...
.....
IS IT A PROPERTY?!
...YES, A PROPERTY CALLED «CONSISTENCY»
WHAT ABOUT «COMPLETENESS»?
UMM... MIRUKA-SAN?
flip flip
A...AH, «COMPLETENESS» IS: FOR ANY STATEMENT A IN THAT FORMAL SYSTEM, UMM...

.....!!
shove
RUSTLE
S-SORRY, WE WILL CONTINUE WITH THE REST NEXT TIME
2ND YEAR— MIRUKA-SAN, PLEASE COME URGENTLY TO THE STUDENT COUNSELING ROOM.
I REPEAT AGAIN—
PUSH
MIRUKA-SAN, ARE YOU OKAY?!
MIRUKA-SAN... HAS SOMETHING HAPPENED TO HER?
LATELY, WITH MATHS, OR HER MOOD, I WOULD SAY... SHE'S KIND OF DIFFERENT FROM USUAL.
WELL .. ACTUALLY... I HEARD IT FOR THE FIRST TIME A FEW DAYS AGO AS WELL.

# 9th period: Miruka-san's Program

IN THE END, YOU'RE FEELING FRUSTRATED AND PITIFUL AGAIN, AREN'T YOU...
EH...?
SAYING THAT IT'S FRUSTRATING, IT'S PITIFUL, WHAT'S THAT GOING TO CHANGE?
EVEN IF YOU GET DEPRESSED, NOTHING IN THE WORLD WILL CHANGE.
I'M...
YOU ARE NTELLI-GENT, THERE SHOULD BE A DESTINATION IN FRONT OF YOU FOR YOU TO HEAD TOWARDS, TOO.

I'M... A KID WHO KEEPS GOING AROUND IN CIRCLES, FOREVER OVER AND OVER AROUND THE SAME SPOT.
...
THERE ARE THINGS THAT ONLY YOU CAN DO, THAT NO-ONE ELSE CAN DO.
stomp
stomp
YOU WILL GET IT SOON, SO DON'T BE DEPRESSED.
...HEY. DON'T BE DEPRESSED.

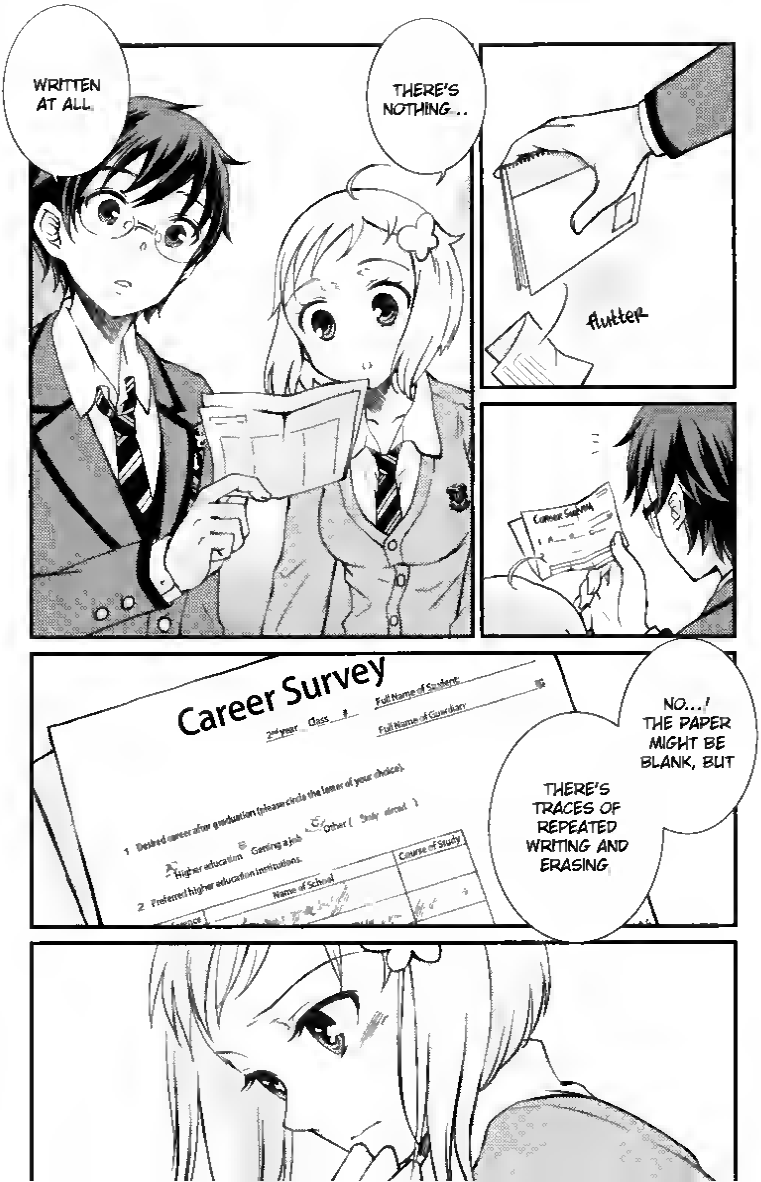
TH... THANK YOU.
REACH
YOU'VE GOT A SCRATCH IT'S GETTING RED.
LET'S DISINFECT IT.
GRAB
.....!

turn
IF IT'S YOU THEN IT'S POSSIBLE.
Eh...
That, just now

WHETHER IT'S OBSTACLES OR PROBLEMS, YOU CAN OVERCOME ANY OF THEM.
I will try to properly accept it.
That's what I can do for now. That's what I should do.

STUDYING ABROAD, YOU SAY. THAT'S AFTER GRADUATION, RIGHT?
YES, STUDYING ABROAD IN AMERICA
S-STUDYING ABROAD?!
2-
T-T-T-THAT'S A SERIOUS MATTER!
Waah
fall
Tōka-chan, calm down
NO, WITH HER SKIPPING A GRADE, SHE WILL BE ADMITTED BY NEXT TERM.
NEXT TERM... YOU SAY, EEEEH?! THAT'S NEXT MONTH, ISN'T IT?!

That seems like it previously
Then, is the student counseling room thing related to her future career?
MIRUKA-SAN IS TAKING A WHILE. IT CAN'T BE HELPED, LET'S GO HOME FIRST.
YES... RIGHT.
HUH? THIS IS MIRUKA-SAN'S NOTEBOOK.
She forgot it?
THAT'S... TO IMAGINE THAT MIRUKA-SAN WILL BE LEAVING...
Sob


WRITTEN AT ALL
THERE'S NOTHING..
flutter
Career Survey
Full Name of Student
Full Name of Guardian
1 Desired career after graduation (please circle the letter of your choice).
A Higher education B Getting a job C Other ( Study abroad )
2 Preferred higher education institutions
Name of School
Course of Study
NO...! THE PAPER MIGHT BE BLANK, BUT
THERE'S TRACES OF REPEATED WRITING AND ERASING

The Miruka-san who has always led us

Even Miruka-san is hesitating...?

Her real feelings...?

THERE'S A FOLD...?
Sorry, Mizuka-san
Let's Challenge Goo... Together!!
Start
Logic quiz
Peano axioms
Mathematical induction
Sets and logic
Limits
Formal system
Formal system of formal system
Diagonal reasoning
Hilbert program
Gödel's incompleteness theorem
Go!
How about trying to make it into a game...
Instead of pretending not to know
Verify what Yuri has already learned!
If Goedel Tetra to realize it herself!
How about getting him to teach this?
Oooh... = 1 ←
Is it difficult at this point... Hmm...
Still thinking
Spring
Summer
Fall
Winter
Gaaah!
3 Months... Is it possible within this time.
A PROGRAM...?

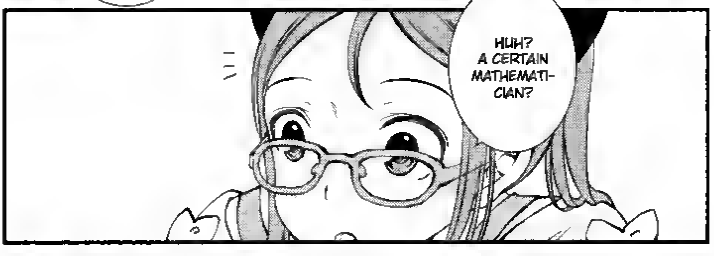
SO HERE IS THE TOPIC FOR TODAY'S LECTURE.
...THERE'S A LITTLE BIT STILL REMAINING.
ONII-CHAN, THERE'S DESSERT!
YOU CAME HERE WITHOUT ME EVEN KNOWING?
Onii-chan!

The three steps of the Hilbert program:
Introducing a formal syst
Expressing mathematics as «formal systems».
Proof of consistency
Proving that the mathe
a formal system is «consi
Pro f of con
Proving that
IT'S BOTHERING YOU? AREN'T WE RELATED? RIGHT, AUNT?
IN THE END, THE CONTINUATION MIRUKA-SAN WANTED TO SAY IS:
THAT'S RIGHT. MAKE SURE TO EAT A LOT, YURI-CHAN.
Even though I keep saying I dislike sweet things
stare
«AT LEAST ONE OF A AND ¬A CAN BE FORMALLY PROVED». THAT'S WHAT THIS PROPERTY MEANS.
«COMPLETENESS» IS: FOR ANY STATEMENT A IN THAT FORMAL SYSTEM,

I'M LISTENING!
Hm!
AND THEN? AND THEN?
WHEN THOSE THREE STEPS ARE ACHIEVED, WHAT HAPPENS NEXT?
hum
Are you listening?
Uwah
Eh, it can't be helped
FOR MATHEMATICS, «THE DARKNESS BEYOND THE REACH OF THE LIGHT OF THE FORMAL SYSTEM WILL BE GONE».
MUNCH
mmm
Ah
here
Huh?

Mathematics
THE RESULT WILL BE A SOLID FOUNDATION OF MATHEMATICS.
TO PUT IT IN OTHER WORDS...
Hilbert program
Introducing a formal system
Expressing mathematics as a «formal system»
Proof of consistency
Proving that the mathematics expressed as a formal system is consistent
THAT'S AMAZING! WHAT AN AMAZING PROGRAM!

BUT, ABOUT THIS PROGRAM, BECAUSE OF A CERTAIN MATHEMATICIAN,
IT BECAME CLEAR THAT THERE ARE THINGS THAT ARE UNATTAINABLE FOR THE ORIGINAL APPROACH



There's no way we can send Miruka-san off with the last topic, [Gödel's incompleteness theorem], unfinished.

—I want Miruka-san to stretch her wings—
YURI-CHAN, SPRING BREAK IS COMING SOON, SO IT'S OKAY.
IT'S OKAY. I'VE ALREADY GOTTEN PERMISSION
I-IT'S KIND OF NERVE-WRACKING.
ONII-CHAN, IS IT REALLY OKAY TO GO IN ON OUR OWN?

I DON'T FEEL AT EASE SINCE MIRUKA-SAN ISN'T HERE
IT'S GOING TO BE ALRIGHT.
Godelscher Unvallstandigkeitssatz
EVERYTHING ON THE PATH WE ARE GOING TO TAKE HAS BEEN ARRANGED FOR US BY MIRUKA-SAN
THAT BOOK IS THE ONE MIRUKA-SAN WAS HOLDING BEFORE...!
Miruka-san's notebook?
nod
fu fu fu
rummage rummage
PUMP
AND HERE TOO, HEAPS OF NOTEPAPER AND PENS. ALL PREPARATIONS COMPLETE!

WELL THEN, LET'S START!
IT'S A CHALLENGE, GÖDEL'S INCOMPLETENESS THEOREM!

# 9th Period

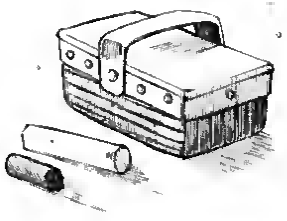
## The Hilbert Program

The Hilbert program, developed by the great mathematician Hilbert, was unfortunately impossible to achieve the way it was originally planned, according to Goedel's incompleteness theorem. However, Goedel neither ruined mathematics nor proved that it's an incomplete thing. We shouldn't be deceived by the word "incomplete". The words "complete" and "incomplete" used in the incompleteness theorem have a strict mathematical meaning, and have a completely different meaning from the words "complete" and "incomplete" that we use in our everyday life.

Hilbert's program stopped halfway, but there's no way that mathematics itself came to a stop. As the fundamental theorem of mathematical logic, Goedel's incompleteness theorem has created a new science called the science of mathematical logic. In other words, the incompleteness theorem is not the end of mathematics. Rather, it has become the start of new mathematics.

«AT LEAST ONE OF A AND ¬A CAN BE FORMALLY PROVED». THAT'S WHAT THIS PROPERTY MEANS.
Shiro
«COMPLETENESS» IS: FOR ANY STATEMENT A IN THAT FORMAL SYSTEM,

Mathematical Girls
Godel s Incompleteness
Theorems



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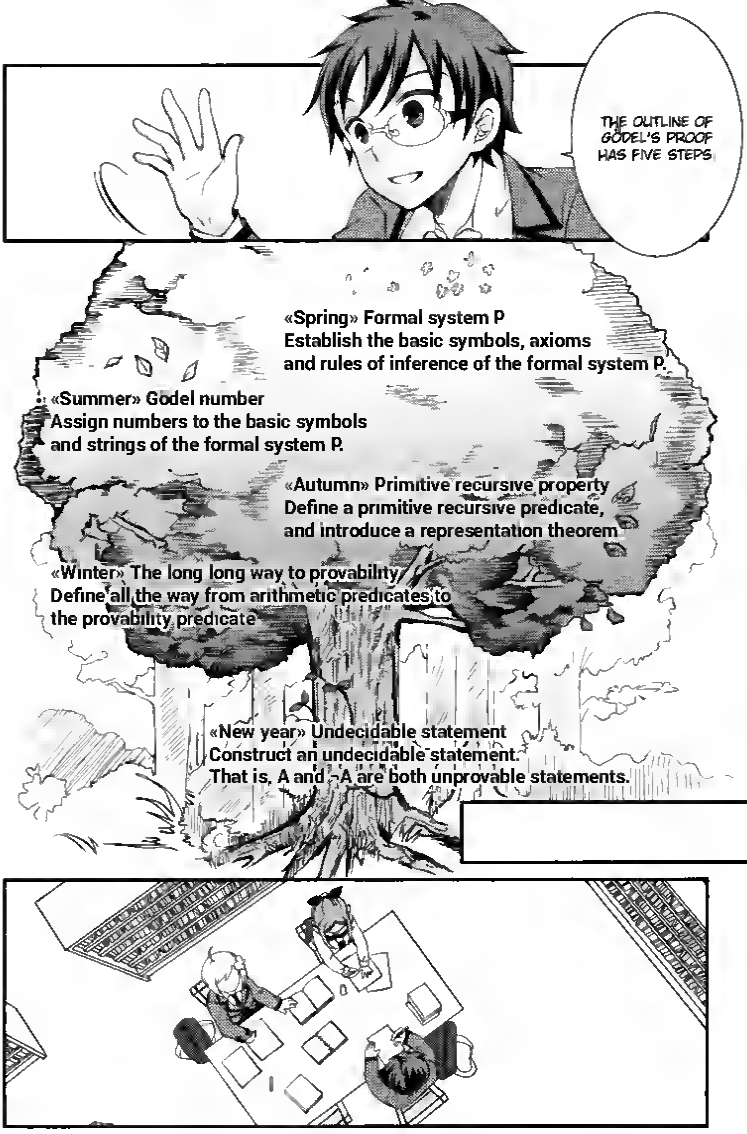
illustrator スらんと

Gödel's first incompleteness theorem:
In any formal system satisfying certain conditions, there exists a statement A, for which both the following hold:
• There is no formal proof of A in that formal system.
• There is no formal proof of ¬A in that formal system.
Gödel's second incompleteness theorem:
In any formal system satisfying certain conditions, no formal proof for a statement that expresses the self-consistency of the system exists.
THERE ARE TWO THEOREMS WHICH TOGETHER ARE CALLED «GOEDEL'S INCOMPLETENESS THEOREMS».
THAT'S... A HEAVY BLOW TO HILBERT'S PROGRAM
FOR A FORMAL SYSTEM THAT SATISFIES CERTAIN CONDITIONS,
NEITHER «COMPLETENESS» NOR «SELF-CONSISTENCY» CAN BE FORMALLY PROVED WITHIN THAT SYSTEM, HUH

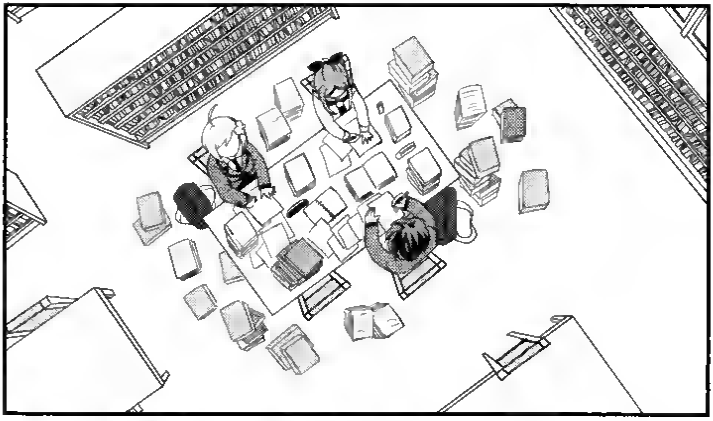

THE OUTLINE OF GÖDEL'S PROOF HAS FIVE STEPS.
«Spring» Formal system P
Establish the basic symbols, axioms and rules of inference of the formal system P.
«Summer» Gödel number
Assign numbers to the basic symbols and strings of the formal system P.
«Autumn» Primitive recursive property
Define a primitive recursive predicate, and introduce a representation theorem.
«Winter» The long long way to provability
Define all the way from arithmetic predicates to the provability predicate
«New year» Undecidable statement
Construct an undecidable statement. That is, A and ¬A are both unprovable statements.

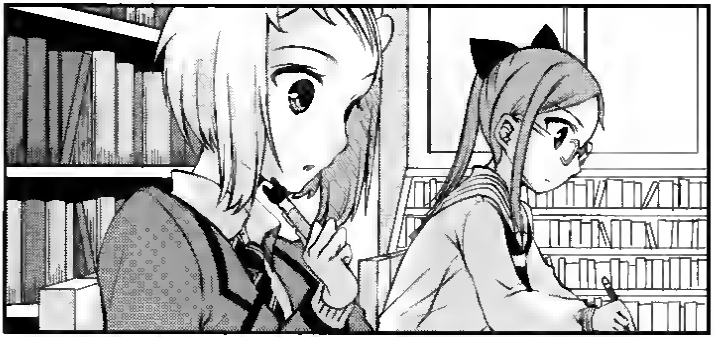
PRIMITIVE RECURSIVE PROPERTY? I WONDER WHAT THIS MEANS.
IT'S THIS PART OF THIS PAGE OF THIS BOOK. FROM HERE WE START «AUTUMN».
AH!
THERE'S AN EXPLANATION WRITTEN IN MIRUKA-SAN'S NOTEBOOK.
Let me see
Where ?

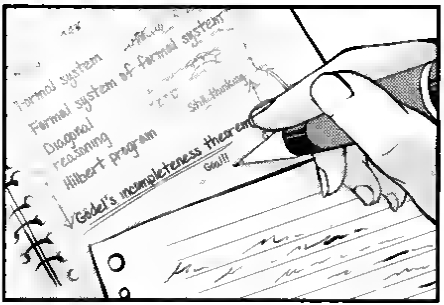
Last lecture Mathematical Girls: Gödel's Incompleteness theorems
55
21
89
8

WOW!
THAT'S EVERYTHING WE'VE BEEN THROUGH UP TO NOW!
WE'VE DONE A LOT TOGETHER.
THE LOGIC QUIZ AT YURI'S CULTURAL FESTIVAL WAS FUN!
AND WE DID THE FORMAL SYSTEM IN THE AMUSEMENT PARK TOGETHER WITH EVERYONE.
SETS AND LOGIC.
THE PEANO AXIOMS FROM THE PRETENDING NOT TO KNOW GAME.
FOR LIMITS, IT WAS THE MATHEMATICS CLASSROOM.
DIAGONAL REASONING.
A FORMAL SYSTEM OF A FORMAL SYSTEM!
MATHEMATICAL INDUCTION.
THE HILBERT PROGRAM
AND SO, THIS WAS THE JOURNEY TOWARDS GÖDEL'S PROOF.

IF THAT'S THE CASE, THEN LET'S TRY WRITING DOWN A COPY OF ALL THIS.
squash
HMM... BUT I CAN'T UNDERSTAND JUST BY READING IT.
WHILE WE WRITE THEM DOWN, LET'S READ ALOUD TO CONFIRM WHAT AND WHERE
THE PARTS WE DON'T REALLY UNDERSTAND ARE
YUP! LET'S DO IT!!
WE STILL HAVE LOTS OF TIME TILL CLOSING TIME AT 4 O'CLOCK. YURI-CHAN, LET'S DO OUR BEST!



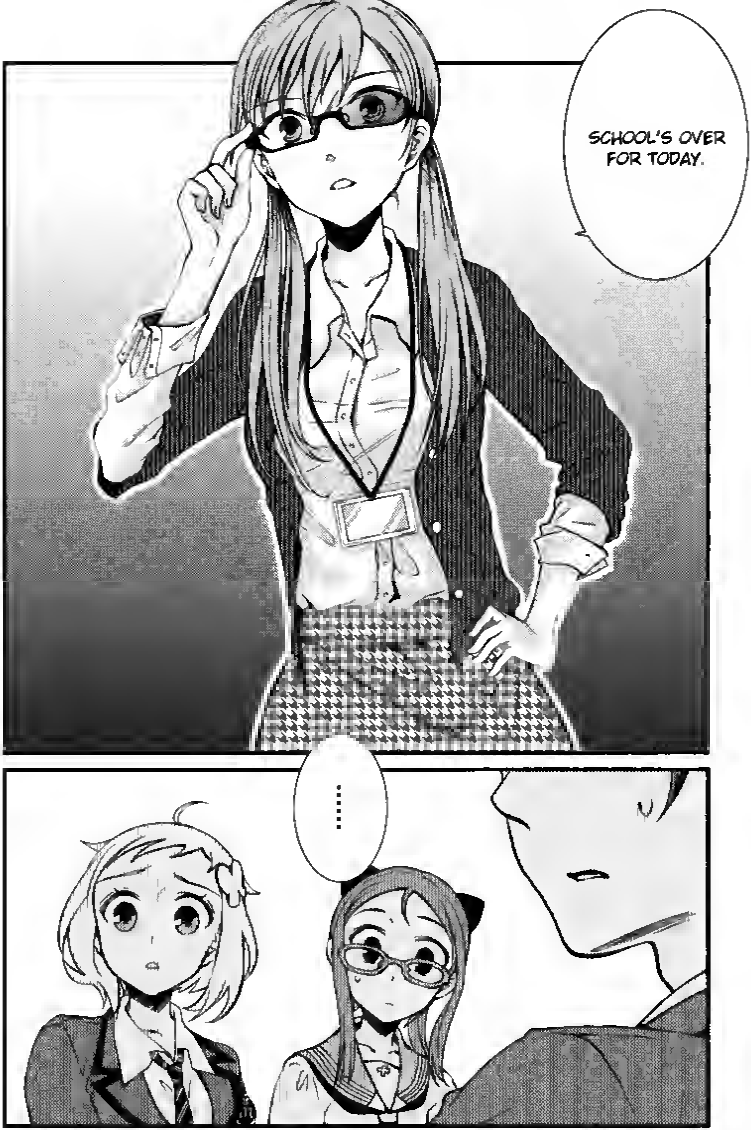



Formal system
Formal system of Formal system
Diagonal reasoning
Hilbert program
Gödel's incompleteness theorem
Gödel
Shit-Hacking

tick
tick
tick
tick
SENPAI, WHAT DOES THIS REFER TO?
TH-THIS... I'VE SEEN IT SOMEWHERE
Umm...

...Oh!
The time...
SEIWO
Hm mm
PA5(P(1)∧
[P(k)⇒
P(k')]
∀k∈N
⇒∀n∈
N[P(n)
DON'T PANIC, TRY TO DECIPHER THEM ONE AT A TIME.
Wait a little
HEY, WHAT'S THIS? ONII-CHAN, I DON'T SEEM TO UNDERSTAND.

STAY CALM...
A LITTLE MORE. JUST A LITTLE MORE...!!
SEIKO
CLICK


SCHOOL'S OVER FOR TODAY.
……

...ALRIGHT
Library
rattle rattle
MIZUTANI-SENSEI IS THE LIBRARIAN AND SHE ANNOUNCES THE END OF THE SCHOOL DAY.
THAT JUST NOW, WAS SHE THE LIBRARY TEACHER?

In the end, perhaps it won't work if Miruka-san is not here
THERE ARE THINGS THAT ONLY YOU CAN DO,
THAT NO-ONE ELSE CAN DO

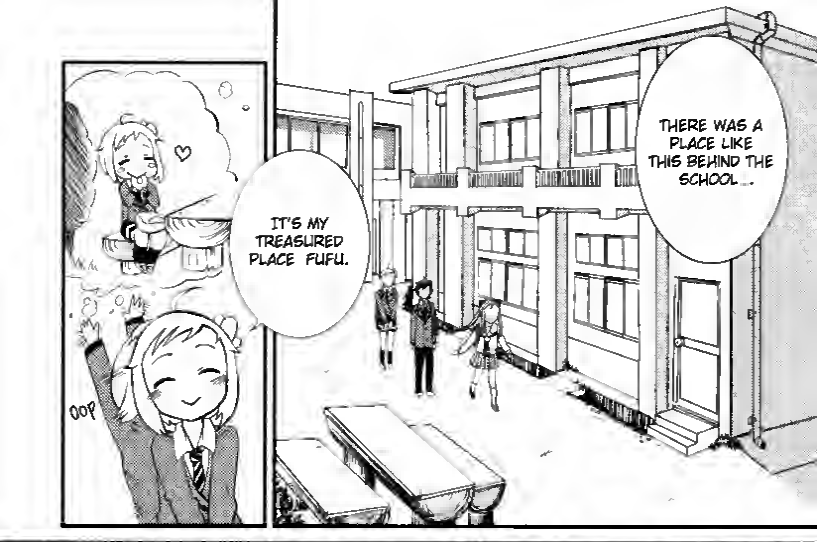
Things
I can
do...

Things that—
—no-one else can do.
I LEARNT THE JOY OF MATHEMATICS FROM SENPAI.
WHETHER IT'S OBSTACLES OR PROBLEMS, YOU CAN OVERCOME ANY OF THEM.
AS EXPECTED, ONII-CHAN IS REALLY GOOD AT TEACHING, NYA. I WISH YOU WERE A TEACHER.
I HOPE THAT MANY PEOPLE ARE, THROUGH SENPAI, ABLE TO KNOW THE JOY OF LEARNING MATHEMATICS.
That's...

YOU TWO, WAIT!!
TEACHING !!!
DON'T GIVE UP, EVERYONE.

Calm down
Let's think
U, UMM.
1 ¬(f(x1) = 0)
2 (f(x1) = f(y1)) → (x1 = y1)
3 X2(0) ∧ ∀x1(X2(x1) → X2(f(x1))) → ∀x1(X2(x1))
FIRST, LET'S RECALL THE HINTS WE'VE RECEIVED FROM MIRUKA-SAN SO FAR.
Onii-chan?
COME TO THINK OF IT, THAT THING EARLIER...
It was bothering me...
I THINK I'VE SEEN A SIMILAR FORM SOMEWHERE BEFORE...
AH!
COULD IT BE THE PEANO AXIOMS THAT WE DID WITH MIRUKA-SAN?!
That's right! That's the spirit!
IF WE PILE UP WHAT WE HAVE LEARNT TOGETHER THUS FAR, WE WILL CERTAINLY UNDERSTAND.

That's right. Miruka-san always sets up many guide-posts on our way.
YURI, THE PART YOU SAID THAT YOU DIDN'T UNDERSTAND A WHILE AGO,
IT MIGHT BE THAT YOU COULD USE THE MATHEMATICAL INDUCTION WAY OF THINKING THERE.
AH!
THAT'S DOMINO TOPPLING, YURI MIGHT HAVE HAD THIS TAUGHT TO HER!
LET'S GO IMMEDIATELY!!
I KNOW A TABLE WE CAN USE EVEN AT A TIME LIKE THIS.
All right.

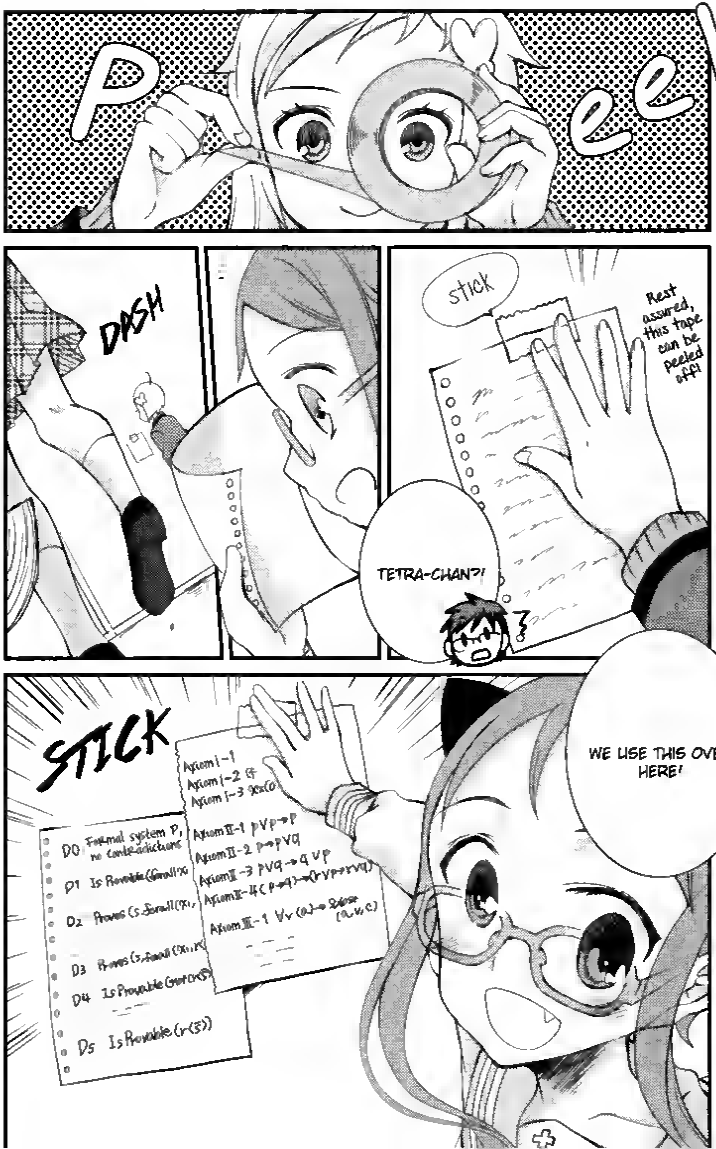

THERE WAS A PLACE LIKE THIS BEHIND THE SCHOOL...
IT'S MY TREASURED PLACE FUFU.
OOP

LET'S ALL WORK TOGETHER AND DO OUR BEST!
STARTING NOW, IT'S THE CRITICAL POINT!
YEAH-!!

NOW THAT I REMEMBER IT, THOSE «CERTAIN CONDITIONS» WHICH KEEP COMING UP...?
THE SYSTEM SHOULD BE CONSISTENT, HAVE A RECURSIVE SET OF AXIOMS, AND CONTAIN PEANO ARITHMETIC
IN OTHER WORDS,
WE HAVE A SYSTEM WHERE WE CAN VERIFY WHETHER ANY STRING CORRESPONDING TO A PROPOSITION ABOUT NATURAL NUMBERS IS A CORRECT FORMAL PROOF WITHOUT CONTRADICTION.
THAT'S WHAT IT MEANS?
VERIFIERS, IS IT . YURI REMEMBERS THOSE TOO.

BECAUSE ALL THE THINGS WE HAVE DONE UP TO NOW, THEY WERE ALL STEPPING STONES TOWARDS THIS PROOF.

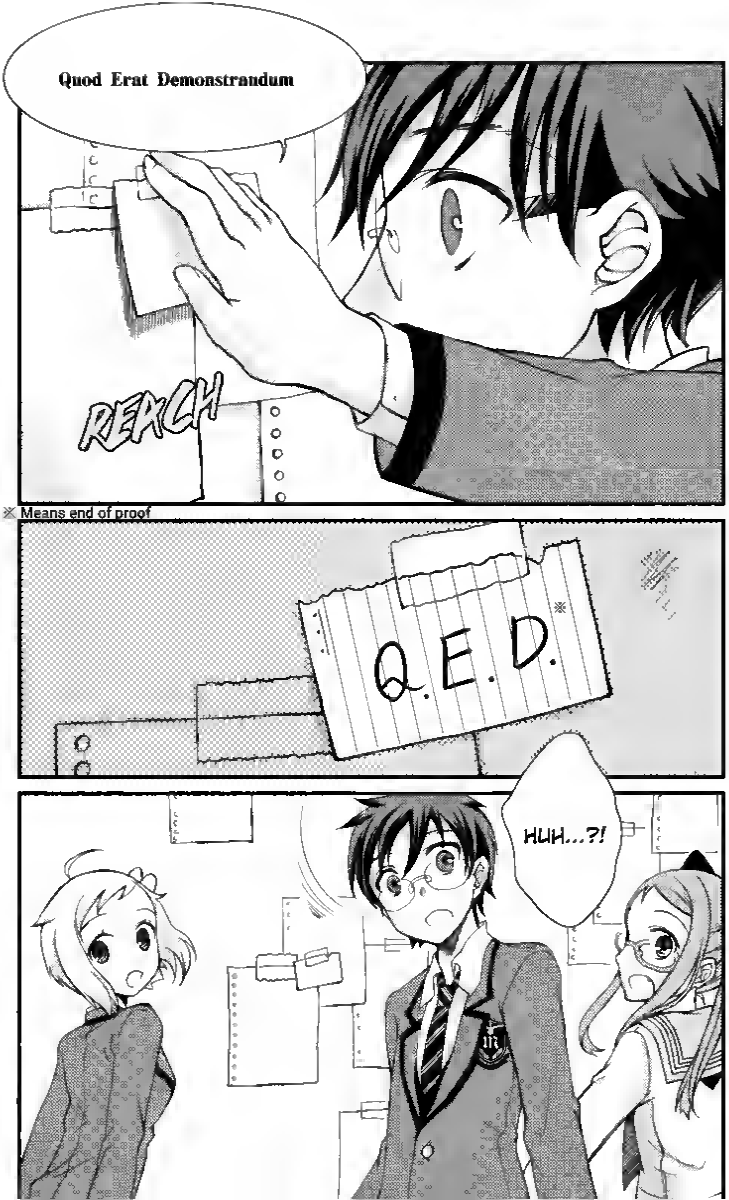
BAM!!
Alright!
WELL THEN, IT'S FINALLY TIME FOR THE PROOF NOW.
WAH! THE DESK'S GOTTEN VERY CROWDED
IT WAS ALL FOR THE SAKE OF THIS...
MIRUKA-SAMA IS AMAZING..!
IT'S ALRIGHT: LET'S MAKE THIS INTO A BIG BLACKBOARD!


PEEL
DASH
STICK
Rest assured, this tape can be peeled off!
TETRA-CHAN?!
STICK
D0 Formal system P, no contradictions
D1 Is Provable(Gödel no
D2 Proves (S-Bonull(
D3 Proves (S-Bonull(
D4 Is Provable(Gödel(
D5 Is Provable(r(S))
Axiom I-1
Axiom I-2
Axiom I-3
Axiom II-1 P∨P→P
Axiom II-2 P→P∨Q
Axiom II-3 P∨Q→Q∨P
Axiom II-4
Axiom III-1
WE USE THIS OVER HERE!

THE FORMAL SYSTEM P IS INCOMPLETE
THE FIRST INCOMPLETENESS THEOREM HAS BEEN PROVED
THE LAST PROOF, THAT'S FOR SENPAI TO WORK OUT, PLEASE

G9 ¬IsProvable(c)
G9 ASSERTS THAT C CAN'T BE FORMALLY PROVED.
THAT IS TO SAY, «A "FORMAL SYSTEM" HAS "NO CONTRADICTIONS"»,
A FORMAL PROOF FOR THE PROPOSITION EXPRESSING THIS DOESN'T EXIST IN THE FORMAL SYSTEM P.

THEREFORE .
THE SECOND INCOMPLETENESS THEOREM HAS BEEN PROVED.

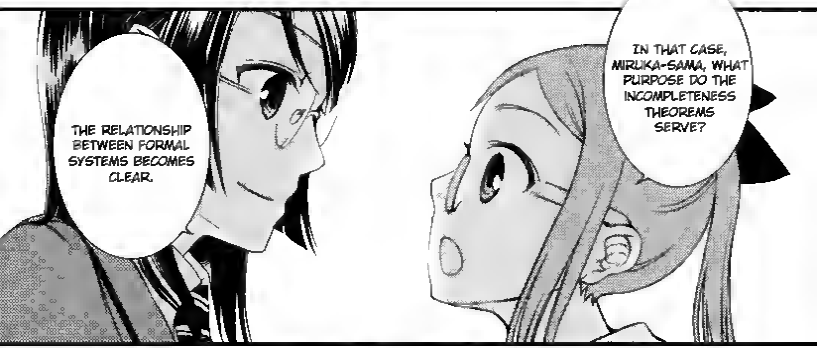

Quod Erat Demonstrandum
REACH
※ Means end of proof
Q.E.D.※
HUH...?!

MI... MIRUKA-SAN?!
YOU WERE SAYING THAT YOU HAD SOME PROCEDURES CONCERNING YOUR VISA TODAY...?
IT'S JUST THAT I CAME ACROSS SUCH A MAGNIFICENT TREE OF MATHEMATICS ON MY WAY BACK FROM THE STAFF ROOM
Staff Room

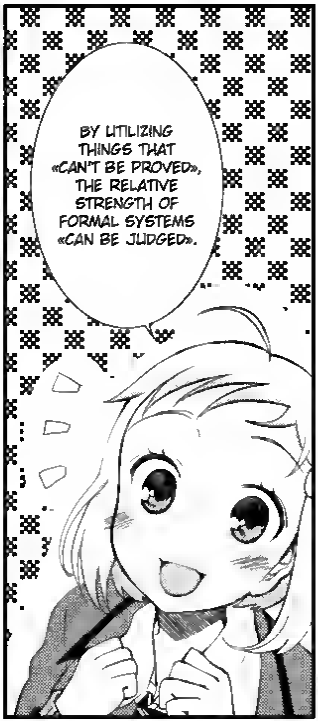
SO, YOU MANAGED TO REACH ALL THE WAY TO THIS, HUH.
G7 Is Provable(c)
G8 Is Provable
G9. ¬ Is Provable(c)
.....
It was my feelings which were beyond solving—
IT'S ALL THANKS TO MIRUKA-SAN THAT WE WERE ABLE TO GET HERE.
WHAT A LOVELY PROOF.
A MAP OF A JOURNEY CALLED GÖDEL.
I believed that if it was you guys, then you could.

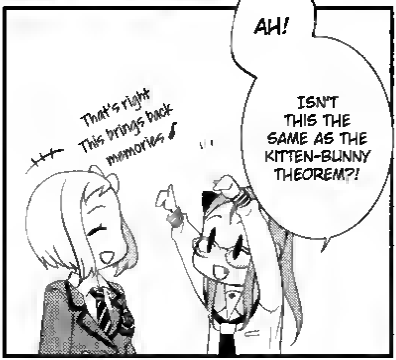
More than anything, the time I did maths together with you guys was fun.
I OUGHT TO ADVANCE FORWARD AS WELL.

BY THE WAY,
Back to our subject
DIDN'T WE END UP PROVING THAT «MATHEMATICS IS INCOMPLETE» WITH THIS?
THAT «INCOMPLETE» IS USED IN A LEXICAL MEANING
Miruka-sama
IT GIVES AN IMPRESSION OF HAVING DEFECTS AND BEING IMPERFECT,
HOWEVER, THIS «INCOMPLETE» IS A FORMAL LOGICAL TERM
WHAT GÖDEL HAS ACTUALLY PROVED IS: FOR A FORMAL SYSTEM THAT SATISFIES CERTAIN CONDITIONS, AMONG THE STATEMENTS THERE IS SOME A,
SUCH THAT NEITHER A OR ¬A CAN EVER BE PROVED.
AND NOT SOMETHING ARGUING WHETHER THE ENTITY CALLED MATHEMATICS HAS DEFECTS OR NOT.

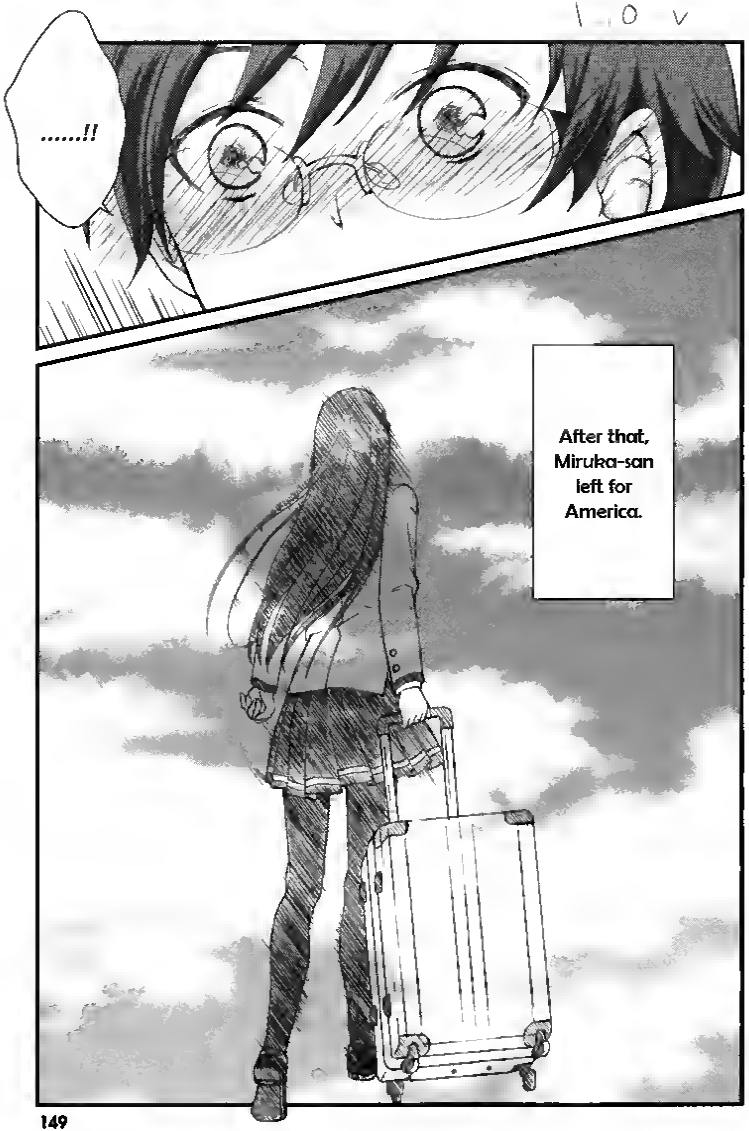

IN THAT CASE, MIRUKA-SAMA, WHAT PURPOSE DO THE INCOMPLETENESS THEOREMS SERVE?
THE RELATIONSHIP BETWEEN FORMAL SYSTEMS BECOMES CLEAR.

THINGS YOU CAN'T DO ARE NOT NECESSARILY WEAKNESSES.
It's totally the usual Miruka-san.


BY UTILIZING THINGS THAT «CAN'T BE PROVED», THE RELATIVE STRENGTH OF FORMAL SYSTEMS «CAN BE JUDGED».


AH!
ISN'T THIS THE SAME AS THE KITTEN-BUNNY THEOREM?!
That's right This brings back memories ♪

Prove
FROM THE NUMERICAL FORMULA, LET'S TRY DERIVING A MESSAGE LIKE THIS.
Is Provable(C)
¬ Is Provable(C)
THEN, HERE'S ONE FOR YOU.
?
G9 ¬ Is Provable(C)


.....!!
After that, Miruka-san left for America.

TL Note: This is a famous graduation song:
https://es.wikipedia.org/wiki/Aogaba_T%C5%8Dtoshi

All of a sudden, my heart is shaken and tears well up spontaneously.
Graduating students, enrolled students— and me as well.
Yuri-chan thank you
Tetra-san your valedictory speech was cool
WOOOOSH
!
Shhhh

Talking together about a word called mathematics,
Girls who shared a time called youth with me—
The End!

Last lecture

# Gödel's incompleteness theorems

To say that a formal system is "complete" means that "for all statements A, a formal proof for either A or ¬A exists." Godel's second incompleteness theorem points out that since for all formal systems satisfying certain conditions, "Statement A exists such that no formal proof for A or for ¬A exists", those systems are incomplete.

HOWEVER, THE MEANING OF THIS «INCOMPLETE» IS A FORMAL LOGICAL TERM.
WHAT GÖDEL HAS ACTUALLY PROVED IS, FOR A FORMAL SYSTEM THAT SATISFIES CERTAIN CONDITIONS AMONG THE STATEMENTS THERE IS SOME A
SUCH THAT NEITHER A OR ¬A CAN EVER BE PROVED.

Although Godel's incompleteness theorem is not difficult to the extent that it's impossible to understand, it is a theorem that ends up easily misunderstood when you don't have knowledge about formal systems.

"Mathematics' making mistakes was proved" or "Humans" ability to reason was denied." There's a lot of people who misunderstand it like this.

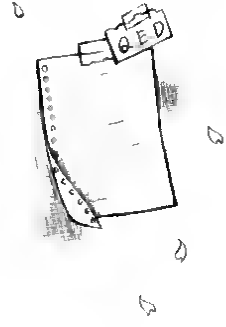
For mathematics, if one doesn't straighten things up precisely and think, one will end up losing one's way and misunderstanding immediately. The "kitten-bunny theorem" or "0.999…=1" which appeared in volume one were also like this. Thinking using one's intuition doesn't work.

If mathematics is studied with respect, it is a field of study that can be enjoyed for a lifetime. The intellect of mankind is gathered there. However, if one thinks using intuition without discipline, and gets puzzled by the meaning of everyday words, it will become incomprehensible.

At a time like that, a maths book, a friend to study together, or a teacher will certainly be of help to you.

Wouldn't you too like to try studying this mathematics which can firmly deal with a marvelous world that can't be grasped by intuition.

Mathematical Girls
Godel's Incompleteness
Theorems



# Mathematical Girls C10

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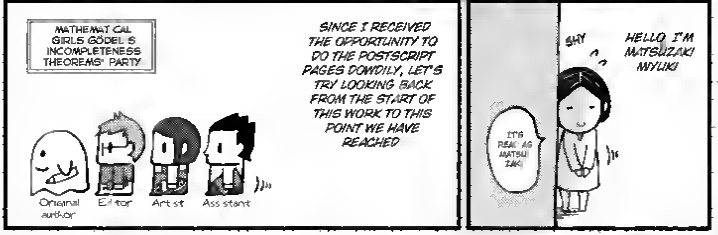
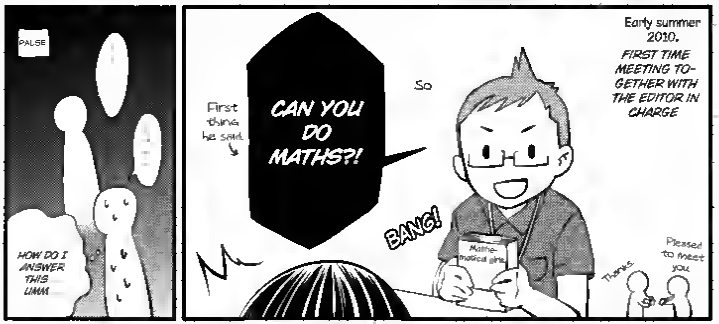
Illustrator ななんと

Postscript manga.

# The making of Mathematical Girls.


MATHEMATICAL GIRLS GÖDEL'S INCOMPLETENESS THEOREMS' PARTY
Original author
Editor
Artist
Assistant
SINCE I RECEIVED THE OPPORTUNITY TO DO THE POSTSCRIPT PAGES DOWDILY, LET'S TRY LOOKING BACK FROM THE START OF THIS WORK TO THIS POINT WE HAVE REACHED
IT'S REALLY MATSUZAKI
SHY
HELLO, I'M MATSUZAKI MYUKI



HONESTLY, IT'S...

**difficult.**

IT'S A SPLENDID MATHEMATICAL BOOK, AFTER ALL.

Some thing I don't know

What's she reading

Uh... Hmm...

FOR THAT REASON I'M EARNESTLY READING THE 3 VOLUMES OF THE ORIGINAL WORK.

The fourth volume is now on sale!!

Even in the middle of the family vacation

FOR THERE TO BE SUCH A FINE BOOK! AND IF I COULD BE THE BRIDGE LINKING READERS AND MATHS!!

I WILL DO IT!

Let me do it please

ME TOO. IF I HAD BEEN READING THIS IN MY SCHOOL DAYS, THEN MAYBE IT WOULD HAVE BEEN DIFFERENT...

BUT HAPPENING ON IT NOW IS ALSO DESTINY.

Part time

Manga

THERE'S MORE THAN THIS TO MATHS OF COURSE. IMPORTANT THINGS, THINGS TO LEARN AND THINGS TO THINK OF. IF YOU KNOW THIS, IT WILL MAKE STUDYING FUN! IT WAS A BOOK FULL OF THIS SHINY STUFF.

HOWEVER, WHEN YOU UNDERSTAND A TOPIC IT FEELS PLEASANT, AND WHEN THOSE CONNECT TOGETHER IT IS FUN!

Now, the labels are full of writing

For some reason, Ay-ay was very popular with some

I wanted to draw more of her paired with Maruka-san!

NOW, THEREFORE, TO CHARACTER CREATION. SINCE THERE WERE NO MANGA ILLUSTRATIONS IN THE ORIGINAL WORK, I HAD THE PRIVILEGE OF MAKING THE DESIGNS THE WAY I LIKE.

Since I had never drawn such a fluffy girl like Tetra-chan it was fun.

UPON PARTING AT THE STATION.
EXTEND
IN SPITE OF THIS BEING THE FIRST MEETING, HE APPROACHED ME AS A FRIEND, AND I WAS ABLE TO HEAR INTIMATE STORIES. IT BECAME AN EDIFYING MEETING FOR THINGS LIKE INSIDE STORIES AND SECRET SETTINGS
AND MIRUKA-SAN WILL PI-AND
TETRA-CHAN IS PI-
THEN, FINALLY, THE FIRST MEETING WITH THE ORIGINAL AUTHOR YUUKI-SENSE!
I SEE-
OH-
PA DUMP PA DUMP
YES! I WILL DO MY BEST!!
TO THE END I RECEIVED WARM ENCOURAGEMENT FROM YUUKI-SENSE!
.....?
THIS WILL CERTAINLY BECOME A FINE WORK!
Night of Gödel
--End of prologue--
WELL THEN FROM HERE WE COME TO THE PRODUCTION OF THIS WORK
FIRST COOPERATION WITH Y-SAN AND CREATION OF THE MATHEMATICAL ELEMENTS OF THE PLOT
THEN CHECKING AND SUPERVISION BY YUUKI-SENSE!
AFTER THAT, ALONG COMICALIZE'S FASCINATING LINES COMPOSING THE STORY FURTHER.

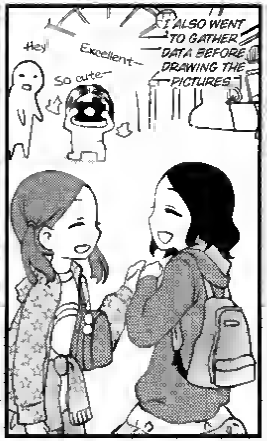
Rather than a preparatory meeting, it was a study meeting.
Earnest victim?
IS THIS CORRECT FOR THIS FORMULA?
SO IT BECOMES LIKE THIS?
AS FOR THAT, IT'S THE FORMULA HERE
NO, DO IT LIKE THIS SO IT'S CORRECT
THAT'S WHY THE PREPARATORY MEETING WAS
FOR THE EXTREMELY DIFFICULT PROBLEM OF GODEL'S INCOMPLETENESS THEOREM TO LEAD TO THE LAST CHAPTER, WE RECONSTRUCT THE MATHEMATICAL PARTS

FIRST PERIOD'S KITTEN-BUNNY THEOREM WAS AN ORIGINAL I THOUGHT UP TOGETHER WITH EDITOR-SAN.
HOW ABOUT EARS! LIKE KITTEN EARS OR BUNNY EARS!!
LET'S MAKE THAT INTO A THEOREM.

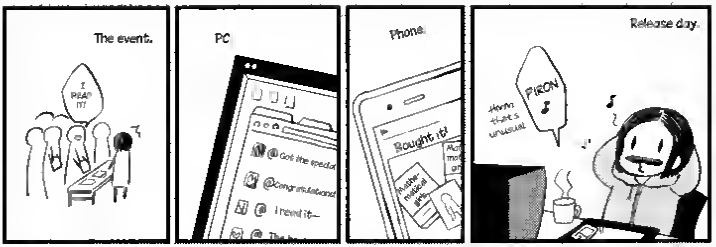
Listen, I realized that the editor-san graduated from U university.

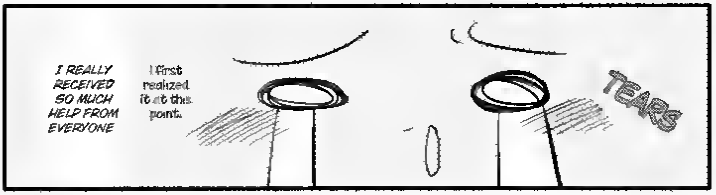
I THINK THAT ARGUING BACK AND FORTH LIKE THIS WAS DISTINCTIVE ABOUT THIS WORK.
AS FOR THAT, HOW ABOUT OO?
I WILL TRY LOOKING IT UP!
THIS PART I WANT TO EXPRESS IT USING AN EASY TO UNDERSTAND ILLUSTRATION...
AND THE STORYBOARD TOO.
The title page is puzzling every time.

Baku's hair is trouble... Cough cough
NOW, DOING THE ROUGH SKETCHES, PEN INKING AND FILLING AND TONES, NOW IT'S FINISHED.
E-
ROUGHLY



FINALLY, THE COMICS WERE COMING OUT!
BUSTLE
FOR REAL?!
IT WILL BE ON SALE IN APRIL
IN THE OSTEOPATHIC CLINIC I RECEIVED TREATMENT AND ENCOURAGEMENT.
IT'S ALRIGHT YOU CAN DO IT!
By the way do you play the drum?
Yes, I...
SHRIND
Daily course of events.
Work.
Tea
Bath.
Hiiikomori
Hwaaa
REPEATING THIS DAY AFTER DAY.
I moved into a house where water could be reheated.
I will try to get an appointment
THAT PERSON ...!!
By IR means? Please.
An Looks like it's the same person.
BOTH ARE BY BAY BRIDGE STUDIO -SAN
Ah, this is really good too~
OH! THIS COVER IS CUTE
WHAT ARE WE GOING TO DO ABOUT THE DESIGNER-SAN?
SMILINGLY, THE IDEAS KEPT COMING SMOOTHLY.
WHAT DO YOU THINK OF THIS?
WITH THE COLORED PAGES LIKE THIS,
LET'S USE THOSE MINI CHARACTERS WHO HAVE NO BASIC LINES.
THEN, IN THE MEETING WITH THE DESIGNER-SAN
She had a portfolio (collection of work).
THEN, VOLUME 1
Release day.
First fanikoubon!
SINCE EDITOR-SAN WAS LOST TOO, THE EDITORIAL DEPARTMENT TOOK A VOTE OR SOMETHING LIKE THAT...!
LATER, DIFFERENT PRINTS OF DESIGN PATTERNS ARRIVED.
Vent Nouveau has really nice colour... How but metal is really cool too~
Cooo~ This is making me lose~
The casual table of contents is really cute!!!




IF WE ARE ABLE TO MEET AGAIN SOMEWHERE, I WILL BE REALLY HAPPY.

TO YOU WHO HAVE THIS BOOK IN YOUR HANDS, BESIDES THEIR YOUTH, I WISH SOMETHING REMAINS IN YOUR HEARTS.

MIRUKA-SAN, TETRA-CHAN, YURI, AND BOKU AS WELL, BEING ABLE TO FIGHT TOGETHER WAS SO MUCH FUN!
THANK YOU!

PLEASE KEEP FOLLOWING, AND ALSO PLEASE TRY READING THE ORIGINAL WORK. THE ENJOYMENT OF MATHEMATICS, THINGS TO LEARN, THING THAT CAN'T BE DRAWN, THERE'S SO MUCH MORE IN THERE.

LIKE THIS, WITH HELP FROM MANY PEOPLE, THIS IS THE COMPLETED "MATHEMATICAL GIRLS" COMIC, GÖDEL'S EDITION.

THEN, GETTING TO THE LAST CHAPTER.

Thank you!

[Mathematical Girls: Gödel's Incompleteness Theorems]
Original author's postscript.

Hello, this is Yuki Hiroshi, the original author.
This work is volume three of the Mathematical Girls series [Mathematical Girls: Gödel's Incompleteness Theorems] which was kindly reorganized into a comical version by Matsuzaki Miyuki-sensei.
I think everyone who has seen this work has been able to feel the enjoyment of friends who love mathematics coming together to study, and challenging even the difficult things.
I'm grateful to Matsuzaki-sensei who put it into such a cheerful and lovely work.
Those who want to experience deeper mathematical content, it would make me happy if you also get your hands on the original work.
Being troubled by mathematics, being troubled by human relations, and being troubled about a career. During youth, trouble never runs out. The characters too are troubled by these things.
However, that's why human life is interesting.
Mathematics is the same. Problems never run out.
The moment you solve a problem, another problem immediately appears.
However, that's why mathematics is interesting.
Always have a fresh heart for mathematics— and then, let's tackle that problem in front of our very eyes.
Mathematics is the accumulation of history.
No individual who can create it all from the start alone exists.
Let's go on, humbly learning from the past and boldly facing the future.
We are bound to encounter friends who are taking the same route.
I wish that everyone, through this work, gets to realize the joy of learning mathematics.

Enjoy!

Mathematical Girls series homepage http://www.hyuki.com/girl/

# The First Incompleteness Theorem

Volandum

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The Gödel series ends on a rather incomplete note, without giving a proof of any of the core results mentioned. We[1] remedy this by providing a proof of the First Incompleteness theorem.

**Theorem 1.** Gödel. *Let $T$ be a consistent theory allowing arithmetic to be defined. Then the language of $T$ contains statements which cannot be proven or proven false.*

## 1 Self-referential sentences in natural language

If we are allowed a quotation operator, we can create functions that interact with their own quotation. For instance, consider the function taking one input "is X when preceded by its own quotation".

Have a play around with this and see what happens when you give it various inputs. Then we can try to see what we get by applying it to its own quotation. We get

"is X when preceded by its own quotation" is X when preceded by its own quotation.

This sentence now describes itself, and says "I am X". This is a hint that if we have a way to turn a formula into something expressible in our language, then we can construct a self-referential expression. What we have is access to Gödel encoding, which is a definable process of encoding an arbitrary formula as a natural number.

Recall also that we have verifiers, or logical functions that can distinguish Gödel codes for the sorts of expressions we may want to look at.

## 2 Applying logical algebra

We will therefore want to apply functions to their own Gödel numbers. Let $f$ be an arbitrary function, and $\Gamma$ an operator taking logical expressions to their Gödel numbers. Consider the operation of going from $f$ to $f(\Gamma(f))$, namely applying a function to its own Gödel number.

[1] Many thanks are due to Nathan Bowler for patiently explaining the logical details.

This operation is just substitution, something we can do mechanically inside the language. In other words there's a definable function $q$ such that for any function $f$ taking a single variable $q(\Gamma(f)) = \Gamma(f(\Gamma(f))$.

Now we wish to write a sentence $T$ claiming that it itself satisfies some predicate $P$. In other words, we should have $T \leftrightarrow P(\Gamma(T))$. We want $T$ to be of the form of some function acting on itself, so let $T = f(\Gamma(f))$.

Substituting this back into our condition gives $f(\Gamma(f)) \leftrightarrow P(\Gamma(f(\Gamma(f))))$. But we know that $\Gamma(f(\Gamma(f))) = q(\Gamma(f))$, so we have $f(\Gamma(f)) \leftrightarrow P(q(\Gamma(f)))$.

This suggests that we try $f = P \circ q$, the composite (i.e. apply $q$, then test for $P$). Let's see if it works.

$$T = f(\Gamma(f)) = P(q(\Gamma(f)) = P(\Gamma(f(\Gamma(f)))) = P(\Gamma(T))$$

In other words, we have a "do yourself" function $q$, and we can combine it with any predicate $P$ to give a function $f$, whose application to itself is a sentence declaring that it satisfies $P$.

## 3 Drawing our conclusions

Now that we can write sentences making claims about themselves, recall that we can write "is provable" - this is equivalent to saying "there exists some number which is the Gödel number of a valid proof whose last line is this result". Thus we can write the statement "this statement is not provable".

If there is no statement which cannot be proven true or false, and so all statements in our system are provable or false, then this statement must call itself false, which is the Liar paradox. Therefore not all statements in our system are provable, and this is an example of a statement which is unprovable in the system.

Suppose that it had in fact been possible to choose a $P$ which says of a statement "this statement is false". Then, as the diagonalisation process works, we have a sentence stating "this sentence is false" - again, the Liar paradox. This shows that it is not possible to write an expression in the language itself for falsity or truth, a result known as Tarski's Undefinability Theorem.

## 4 Playing with models

The Second Incompleteness Theorem is a bit more specific: it says that a system cannot prove its own consistency. That means that a system is consistent with the statement saying that it is inconsistent. For instance, Peano arithmetic is consistent with the statement "Peano arithmetic is inconsistent." We can therefore find a model of the combined theory, and look for a witness of the inconsistency of PA (a number coding for a proof of its inconsistency, say), which will be a nonstandard natural number.

# Mathematical Girls

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